Line Integrals

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Outline

Line Integrals of Scalar Functions

Line Integrals of Scalar Functions

Let $f(x, y, z)$ be a function defined on a region containing a curve C parametrized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for t in $[a, b]$.

Choose a partition $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. This gives a partition of C by the points $P_k = (x(t_k), y(t_k), z(t_k))$.

In the kth subarc of C, choose a point (x_k, y_k, z_k). Let Δs_k be the length of the kth subarc of C.

Then we can form the Riemann sum

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k.$$

Line Integrals of Scalar Functions

If we take the limit as the largest of the Δs_k goes to zero, we get the **line integral of** $f(x, y, z)$ **over** C:

$$\int_C f(x, y, z)\, ds = \lim_{\max_k \{\Delta s_k\} \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k.$$

How to Evaluate a Line Integral

How to Evaluate a Line Integral

You evaluate a line integral by parametrizing the curve C by an interval $[a, b]$ and using that parametrization to pull the line integral back to an integral on $[a, b]$, which gives us a Calculus 1 problem.

How to Evaluate a Line Integral

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad a \leq t \leq b$$

2. Evaluate the integral

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\,\frac{ds}{dt}\,dt.$$

I'll remind you that $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$.

How to Evaluate a Line Integral

Theorem 6.4: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

if C is a planar curve and f is a function of three variables.

How to Evaluate a Line Integral

Theorem 6.4: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$. Then

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

if C is a planar curve and f is a function of two variables.

Example

Example 1

Example

Evaluate $\int_C (x + y)\, ds$ where C is the straight-line segment $x = t$, $y = (1 - t)$, $z = 0$, from $(0, 1, 0)$ to $(1, 0, 0)$.

Example 1

Solution

The parametrization for the curve is given to us. We note that the point $(0, 1, 0)$ corresponds to $t = 0$ and the point $(1, 0, 0)$ corresponds to $t = 1$. We compute ds/dt:

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ &= \sqrt{(1)^2 + (-1)^2 + (0)^2} \\ &= \sqrt{2}. \end{aligned}$$

Example 1

Solution (cont.)

Now, we use the parametrization to pull the line integral back to an integral on the interval $[0, 1]$:

$$\begin{aligned}\int_C (x+y)\,ds &= \int_a^b [x(t)+y(t)]\,\frac{ds}{dt}\,dt\\ &= \int_0^1 [t+(1-t)]\,\sqrt{2}\,dt\\ &= \int_0^1 \sqrt{2}\,dt\\ &= \sqrt{2}t\Big|_0^1 = \sqrt{2}-0=\sqrt{2}.\end{aligned}$$

A Comment on Example 1

As a comment, we remark that on the line segment given, $x + y = 1$, so what we have computed in the last example is

$$\int_C 1\, ds$$

which is nothing but the length of the line segment from $(0, 1, 0)$ to $(1, 0, 0)$.

The distance formula shows us our answer is correct.

Independence of Parameterization

Independence of Parameterization

Changing the parameterization does not change the value of the line integral.

Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Additivity

Additivity

If a curve C is made up by joining smooth curves C_1, C_2, $\ldots$, C_n, then

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_n} f\, ds.$$

Examples

Example 2

Example

Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ given by

$$\begin{aligned} C_1 &: \mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}, \quad 0 \le t \le 1 \\ C_2 &: \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 1 \end{aligned}$$

See the sketch of the curve on the next slide.

Example 2

Example

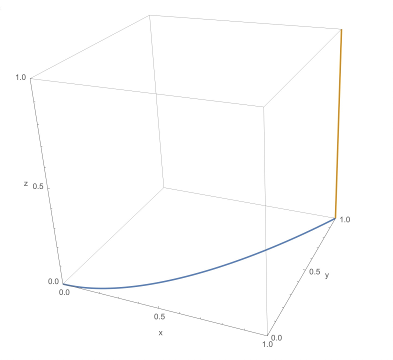

Figure: Sketch of Path C

Example 2

Solution

We first compute the integral over C_1. First, we have $ds/dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$.

$$\begin{aligned} \int_{C_1} (x + \sqrt{y} - z^2)\, ds &= \int_0^1 (t + \sqrt{t^2} - 0^2)\, \sqrt{1+4t^2}\, dt \\ &= \int_0^1 2t\sqrt{1+4t^2}\, dt \\ &= \frac{1}{6}(1+4t^2)^{3/2}\Big|_0^1 = \frac{1}{6}\left(5^{3/2} - 1\right). \end{aligned}$$

Example 2

Solution (cont.)

We next compute the integral over C_2. Here, we have $ds/dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1.$

$$\begin{aligned}\int_{C_2} (x + \sqrt{y} - z^2)\, ds &= \int_0^1 (2 - t^2) \cdot (1)\, dt \\ &= \int_0^1 (2 - t^2)\, dt \\ &= 2t - \frac{1}{3}t^3 \Big|_0^1 = \frac{5}{3}.\end{aligned}$$

Example 2

Solution (cont.)

Finally

$$\begin{aligned}\int_C (x+\sqrt{y} - z^2)\, ds &\\ &= \int_{C_1} (x + \sqrt{y} - z^2)\, ds + \int_{C_2} (x + \sqrt{y} - z^2)\, ds \\ &= \frac{1}{6}\left(5^{3/2} - 1\right) + \frac{5}{3} \\ &= \frac{5\sqrt{5}}{6} + \frac{3}{2}.\end{aligned}$$

Example 3

Example

Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ given by

$$\begin{aligned} C_1 &: \mathbf{r}(t) = t\,\mathbf{k}, \quad 0 \le t \le 1 \\ C_2 &: \mathbf{r}(t) = t\,\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1 \\ C_3 &: \mathbf{r}(t) = t\,\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1 \end{aligned}$$

A sketch of this path appears on the next slide.

Example 3

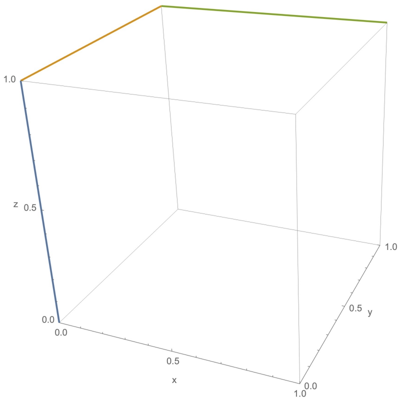

Figure: Sketch of Path C

Example 3

Solution

We first compute the integral over C_1. First, we have $ds/dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(1)^2 + (0)^2 + (0)^2} = 1.$

$$\begin{aligned} \int_{C_1} (x + \sqrt{y} - z^2)\, ds &= \int_0^1 -t^2\, dt \\ &= -\frac{1}{3}t^3\bigg|_0^1 = -\frac{1}{3}. \end{aligned}$$

Example 3

Solution (cont.)

We next compute the integral over C_2. First, we have $ds/dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(1)^2 + (0)^2 + (0)^2} = 1$.

$$\begin{aligned}\int_{C_2} (x + \sqrt{y} - z^2)\, ds &= \int_0^1 \sqrt{t} - 1\, dt \\ &= \frac{2}{3}t^{3/2} - t\Big|_0^1 = -\frac{1}{3}.\end{aligned}$$

Example 3

Solution (cont.)

We finally compute the integral over C_3. Here, we have $ds/dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(1)^2 + (0)^2 + (0)^2} = 1.$

$$\begin{aligned} \int_{C_3} (x + \sqrt{y} - z^2)\, ds &= \int_0^1 t\, dt \\ &= \frac{1}{2} t^2 \Big|_0^1 = \frac{1}{2}. \end{aligned}$$

Example 3

Solution (cont.)

Finally

$$\begin{aligned}\int_C (x+\sqrt{y}-z^2)\,ds &= \int_{C_1} (x+\sqrt{y}-z^2)\,ds \\ &\quad + \int_{C_2} (x+\sqrt{y}-z^2)\,ds \\ &\qquad + \int_{C_3} (x+\sqrt{y}-z^2)\,ds \\ &= -\frac{1}{3} - \frac{1}{3} + \frac{1}{2} = -\frac{1}{6}.\end{aligned}$$

Calculating Arc Length

Example 3

Example

Find the length of a wire with parameterization

$$\mathbf{r}(t) = \langle 3t + 1, 4 - 2t, 5 + 2t \rangle, \quad 0 \leq t \leq 4.$$

Example 3

Solution

We compute

$$
\begin{aligned}
s &= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^4 \sqrt{(3)^2 + (-2)^2 + (2)^2}\, dt \\
&= \int_0^4 \sqrt{9+4+4}\, dt = \int_0^4 \sqrt{17}\, dt \\
&= \sqrt{17}t\Big|_0^4 = 4\sqrt{17}.
\end{aligned}
$$

Vector Line Integrals

Vector Line Integrals

How do we find the work done by a force **F** in moving a particle along a curve C?

The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an **orientation of a curve**. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. Such a curve is called an **oriented curve**.

If a curve begins and ends at the same point, it is called a **closed curve**.

Vector Line Integrals

Let $\mathbf{r}(t)$ be a parameterization of C for $a \leq t \leq b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C.

Choose a partition for the parameter interval $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

Vector Line Integrals

Let P_i denote the endpoint of the vector $\mathbf{r}(t_i)$, $1 \leq i \leq n$. The points

$$\mathbf{r}(a) = P_0 < P_1 < \cdots < P_{n-1} < P_n = \mathbf{r}(b)$$

form a partition of C.

Let Δs_i be the length of C between P_{i-1} and P_i.

For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then the endpoint $\mathbf{r}(t_i^*)$, which we call P_i^*, is a point on C between P_{i-1} and P_i.

Vector Line Integrals

If Δs_i is small then as the particle moves from P_{i-1} and P_i along C, it moves approximately in the direction of $\mathbf{T}(P_i^*)$ the unit tangent vector at P_i^*. Then the work done by the force vector field in moving the particle from P_{i-1} and P_i is $\mathbf{F}(P_i^*) \cdot (\Delta s_i \mathbf{T}(P_i))$, so the total work done along C is

$$\sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot (\Delta s_i \mathbf{T}(P_i^*)) = \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*) \, \Delta s_i.$$

Vector Line Integrals

We recognize the sum

$$\sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*)\, \Delta s_i.$$

as a Riemann sum approximating the integral

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

This is the work done by the force **F** in moving the particle along C.

This is the **vector line integral** of vector field **F** along oriented smooth curve C.

Vector Line Integrals

To have a more usable formula, we note that $\mathbf{T} = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$ and $ds = \|\mathbf{r}'(t)\|\, dt$. So, we have

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Because of this equation, we often use the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

Example

Example 4

Example

Evaluate the vector line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ and the curve C with parametrization $\langle t, t^2 \rangle$ for $0 \leq t \leq 2$.

Example 4

Solution

We compute

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^2 \mathbf{F}(t,t^2)\cdot\langle 1,2t\rangle\,dt\\ &= \int_0^2 \langle t,t^2\rangle\cdot\langle 1,2t\rangle\,dt\\ &= \int_0^2 2t^3+t\,dt = \frac{1}{2}t^4+\frac{1}{2}t^2\bigg|_0^2\\ &= \left(\frac{1}{2}(2)^4+\frac{1}{2}(2)^2\right) = 10.\end{aligned}$$

Vector Line Integrals

If $\mathbf{F} = \langle P, Q, R \rangle$ and we think of $d\mathbf{r} = \langle dx, dy, dz \rangle$, then

$$\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle,$$

and we can write

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C P\,dx + Q\,dy + R\,dz \\ &= \int \left(P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} + R(\mathbf{r}(t))\frac{dz}{dt} \right) dt. \end{aligned}$$

Vector Line Integrals

Theorem 6.5: Properties of Vector Line Integrals

Let **F** and **G** be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$
2. $\int_C k\,\mathbf{F} \cdot d\mathbf{r} = k \int_C \mathbf{F} \cdot d\mathbf{r}$
3. $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$
4. If $C_1, \ldots, C_n$ is a chain of piecewise smooth curves where the end of one is the beginning of the next, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{n} \int_{C_i} \mathbf{F} \cdot d\mathbf{s}.$$

Applications of Line Integrals

Applications of Line Integrals

Calculating the Mass of a Wire

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x, y, z)$. We can calculate the total mass of the wire using the scalar line integral

$$\int_C \rho(x, y, z)\, ds.$$

Applications of Line Integrals

Calculating Work

Suppose a curve C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$. The work required to move an object in a vector field $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz. \end{aligned}$$

Applications of Line Integrals

Flux and Circulation

Let C be a plane curve representing a membrane across which fluid flows and let **F** be a vector field in the plane representing the velocity field of the fluid.

The flux of the fluid across the membrane is then given by

$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds,$$

where **N** is the unit normal vector.

Applications of Line Integrals

Flux and Circulation

More specifically, let C be a plane curve parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$.

Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$ be the vector that is normal to C at the endpoint of $\mathbf{r}(t)$ and points to the right as we traverse C in the positive direction.

Then

$$\mathbf{N}(t) = \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}$$

is the unit normal vector to C at the endpoint of $\mathbf{r}(t)$ that points to the right as we traverse C.

Applications of Line Integrals

Definition

The **flux** of **F** across C is the line integral

$$\int_C \mathbf{F}(t) \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\, ds.$$

Applications of Line Integrals

Theorem 6.6: Calculating Flux across a Curve

Let **F** be a vector field and let C be a smooth curve with parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The flux of **F** across C is

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t)\, dt.$$

Applications of Line Integrals

Definition

Let $\mathbf{F}$ be a vector field and let C be a smooth simple closed curve with parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The **circulation** of $\mathbf{F}$ along C is

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{dr}.$$