Vector Fields

William M. Faucette

University of West Georgia

Fall 2025

Outline

Vector Fields

Vector Fields

A vector field is a function which takes in a point and produces a vector—either in the plane or in space (for our purposes).

Vector Fields

Definition

A vector field $\mathbf{F}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.

A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. Once again, the subset D is the domain of the vector field.

Drawing a Vector Field

Drawing a Vector Field

The typical way of graphically representing a vector field is by placing the vector $\mathbf{F}(x, y)$ with its tail at the point (x, y).

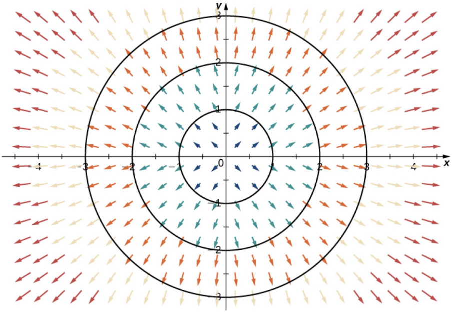

Figure: Vector Field $\mathbf{F}(x, y) = \frac{1}{2}x\,\mathbf{i} + \frac{1}{2}x\,\mathbf{j}$

Drawing a Vector Field

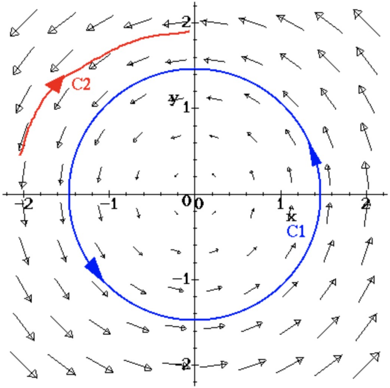

Figure: Vector Field $\mathbf{F}(r, \theta) = r \sin \theta \, \mathbf{i} + r \cos \theta \, \mathbf{j}$

Another type of vector field is a **rotational field**.

Drawing a Vector Field

A unit vector field assigns a unit vector to each point.

The vector field

$$\mathbf{F}(x, y) = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, -\frac{x}{\sqrt{x^2 + y^2}} \right\rangle$$

is a rotational unit vector field.

Vector Fields in $\mathbb{R}^3$

Vector Fields in $\mathbb{R}^3$

A vector field in $\mathbb{R}^3$ is given by

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

or

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}.$$

Vector Fields in $\mathbb{R}^3$

Vector fields in $\mathbb{R}^3$ are more difficult to sketch because they have to be sketched in three dimensions.

The easiest way to do this is on a three-dimensional coordinate system with perspective on the third dimension.

Gradient Fields

Gradient Fields

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved.

Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Gradient Fields

Recall, if $f(x, y)$ is a scalar function having both first partial derivatives, then

$$\text{grad } f = \nabla f = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

Likewise, if $f(x, y, z)$ is a scalar function having both first partial derivatives, then

$$\text{grad } f = \nabla f = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}.$$

Gradient Fields

Definition

A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a **gradient field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

Example

Example 1

Example

Use technology to plot the gradient vector field of $f(x, y) = \sin x \cos y$.

Example 1

Solution

First, we compute grad f:

$$\begin{aligned} \text{grad } f &= f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} \\ &= \cos x \cos y\,\mathbf{i} - \sin x \sin y\,\mathbf{j}. \end{aligned}$$

Example 1

Solution (cont.)

Now, we can use Mathematica to sketch the vector field. The commands are:

$$\text{VectorPlot}[\{\text{Cos}[x]\text{Cos}[y], -\text{Sin}[x]\text{Sin}[y]\}, \{x, -3, 3\}, \{y, -3, 3\}].$$

Remember you have to press "shift-enter" to execute a command in Mathematica.

Example 1

Solution (cont.)

The sketch we get is this:

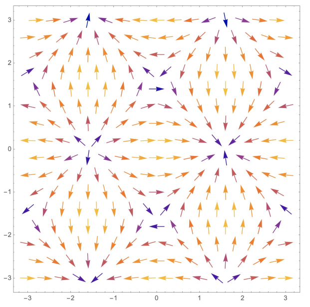

Figure: Vector Field $\mathbf{F}(x, y) = \cos x \cos y\,\mathbf{i} - \sin x \sin y\,\mathbf{j}$

Potential Functions

Potential Functions

As we learned earlier, a vector field $\mathbf{F}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that grad $f = \mathbf{F}$.

In this situation, f is called a **potential function** for $\mathbf{F}$.

Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called conservative is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

Example

Example 2

Example

Verify that $f(x, y) = x^3y^2 + x$ is a potential function for velocity field

$$\mathbf{v}(x, y) = \left\langle 3x^2y^2 + 1, 2x^3y \right\rangle$$

Example 2

Solution

We start with $f(x, y) = x^3y^2 + x$ and compute grad f.

$$f_x(x, y) = 3x^2y^2 + 1$$
$$f_y(x, y) = 2x^3y.$$

Comparing components, we see that $\nabla f = \mathbf{v}$.

Potential Functions

The following theorem tells us that potential functions for a given vector field are unique up to an additive constant.

Theorem 6.1: Uniqueness of Potential Functions

Let $\mathbf{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\nabla f = \nabla g = \mathbf{F}$ Then, there is a constant C such that $f = g + C$.

Potential Functions

Conservative vector fields also have a special property called the *cross-partial property*. This property helps test whether a given vector field is conservative.

This property is stated on the next two slides: one for vector fields in the plane and one for vector fields in the space.

Potential Functions

Theorem 6.2: The Cross-Partial Property of Conservative Vector Fields

Let $\mathbf{F}$ be a vector field in two dimensions such that the component functions of $\mathbf{F}$ have continuous second order mixed-partial derivatives on the domain of $\mathbf{F}$.

If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Potential Functions

Theorem 6.2: The Cross-Partial Property of Conservative Vector Fields

Let $\mathbf{F}$ be a vector field in three dimensions such that the component functions of $\mathbf{F}$ have continuous second order mixed-partial derivatives on the domain of $\mathbf{F}$.

If $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

Potential Functions

You can show a vector field is not conservative by showing it does not have the cross-partial property.

An Important Warning

An Important Warning

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if **F** is conservative, then **F** has the cross-partial property.

The theorem does **NOT** say that, if **F** has the cross-partial property, then **F** is conservative.

This statement is the converse of the first statement and it is false.

An Important Warning

In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

An Important Warning

For example, consider vector field

$$\mathbf{F}(x, y) = \left\langle \frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2} \right\rangle.$$

This field has the cross-partial property, so it is tempting to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem.

We learn later how to conclude that **F** is not conservative.