Change of Variables in Multiple Integrals

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Outline

Substitutions in Multiple Integrals

Substitutions in Multiple Integrals

Let G be a region in the uv-plane and let there be change of coordinates given by

$$x = x(u, v), \quad y = y(u, v)$$

The image of G under this change of coordinates is a region R in the xy-plane.

If we're going to change coordinates, we need to compute how the change of coordinates changes the area element.

The Jacobian

The Jacobian

Definition

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = x(u, v)$, $y = y(u, v)$ is

$$J(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

The Jacobian determinant can also be denoted by

$$\frac{\partial(x, y)}{\partial(u, v)}.$$

Example

Example 1

Example

Solve the system

$$u = x + 2y, \quad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

Example 1

Solution

First, we solve the system for x and y using a bit of algebra:

$$x = \frac{1}{3}(u + 2v)$$
$$y = \frac{1}{3}(u - v).$$

Then we compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} = -\frac{1}{3}.$$

Substitutions in Double Integrals

Substitutions in Double Integrals

Theorem (Substitutions in Double Integrals)

Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = x(u, v)$, $y = y(u, v)$, which is assumed to be one-to-one on the interior of G.

If the functions x and y have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

Substitutions in Double Integrals

Your book says the derivation of this formula properly belongs in a course in advanced calculus. Horse hockey!

A change of $\langle \Delta u, 0 \rangle$ has as image the tangent vector $\left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u$.
A change of $\langle 0, \Delta v \rangle$ has as image the tangent vector $\left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v$.

Substitutions in Double Integrals

The area of the parallelogram spanned by the vectors

$$\left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u \quad \text{and} \quad \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v$$

is the absolute value of the length of the cross product of these two vectors (after you put them into 3-space).

This is exactly $|J(u, v)|\, \Delta u\, \Delta v$.

Substitutions in Double Integrals

Let's do the computation:

$$\begin{aligned}
\Delta \mathbf{r}_u \times \Delta \mathbf{r}_v &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{pmatrix} \\
&= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u \, \Delta v \, \mathbf{k} \\
&= \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \Delta u \, \Delta v \, \mathbf{k} \\
&= J(u,v) \, \Delta u \, \Delta v \, \mathbf{k}.
\end{aligned}$$

Substitutions in Double Integrals

Then we have

$$\|\Delta \mathbf{r}_u \times \Delta \mathbf{r}_v\| = |J(u, v)| \; \Delta u \, \Delta v.$$

So, we see the adjusted area element in uv-space is

$$|J(u, v)| \; du \, dv.$$

Examples

Example 2

Example

Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\, dy.$$

Example 2

Solution

The transformation is given by $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$.

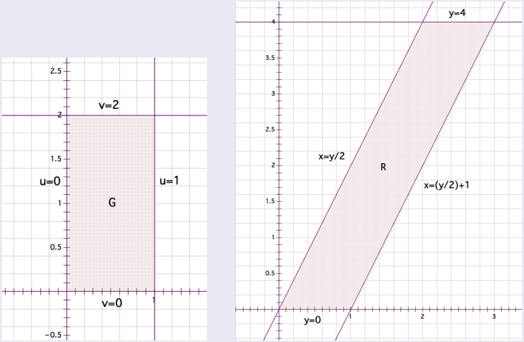
Let's start by sketching the region R in the xy-plane and the region G in the uv-plane.

- The curve $y = 0$ becomes $v = 0$.
- The curve $y = 4$ becomes $v = 2$.
- The line $x = y/2$ becomes $u = 0$.
- The line $x = (y/2) + 1$ becomes $u = 1$.

Example 2

Solution (cont.)

Figure: Sketches of G and R



Example 2

Solution (cont.)

Now we change variables in the integrand. Note that $x = u + v$ and $y = 2v$.

$$\frac{2x - y}{2} = \frac{2(u + v) - 2v}{2} = u.$$

Yes, we already knew this.

Example 2

Solution (cont.)

Now, we compute the Jacobian:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2.$$

Example 2

Solution (cont.)

Now, we apply the change of variables formula.

$$\begin{aligned}\iint_R \frac{2x-y}{2}\,dx\,dy &= \iint_G u\cdot 2\,du\,dv\\ &= \int_0^2\int_0^1 2u\,du\,dv\\ &= \int_0^2 u^2\Big|_0^1\,dv = \int_0^2 1\,dv\\ &= v\Big|_0^2 = 2.\end{aligned}$$

Example 3

Example

Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$, and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) \, dx \, dy$$

Example 3

Solution

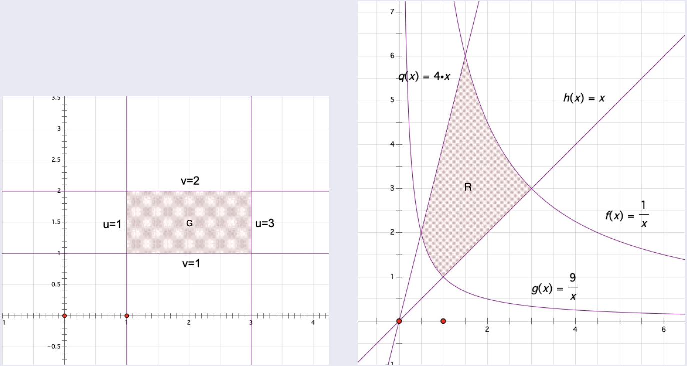
The transformation is given by $x = u/v$, $y = uv$.
Let's start by sketching the region R in the xy-plane and the region G in the uv-plane.

- The curve $xy = 1$ becomes $u^2 = 1$, or $u = 1$.
- The curve $xy = 9$ becomes $u^2 = 9$, or $u = 3$.
- The line $y = x$ becomes $v^2 = 1$, or $v = 1$.
- The line $y = 4x$ becomes $v^2 = 4$, or $v = 2$.

Example 3

Solution (cont.)

Figure: Sketches of G and R



Example 3

Solution (cont.)

Now we change variables in the integrand.

$$\sqrt{\frac{y}{x}} + \sqrt{xy} = \sqrt{\frac{uv}{u/v}} + \sqrt{\frac{u}{v}uv} = v + u.$$

Example 3

Solution (cont.)

Now, we compute the Jacobian:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$
$$= \frac{u}{v} - \left(-\frac{u}{v}\right) = \frac{2u}{v}.$$

Example 3

Solution (cont.)

Now, we apply the change of variables formula.

$$\begin{aligned}
\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right)\,dx\,dy &= \iint_G (u+v)\cdot\left|\frac{2u}{v}\right|\,du\,dv \\
&= \int_1^2 \int_1^3 \left(\frac{2u^2}{v} + 2u\right)\,du\,dv \\
&= \int_1^2 \left.\left(\frac{2u^3}{3v} + u^2\right)\right|_1^3\,dv \\
&= \int_1^2 \left(\frac{52}{3v} + 8\right)\,dv
\end{aligned}$$

Example 3

Solution (cont.)

Continuing,

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right)\, dx\, dy = \int_1^2 \left(\frac{52}{3v} + 8\right)\, dv$$

$$= \frac{52}{3}\ln v + 8v\Big|_1^2$$

$$= 8 + \frac{52\ln(2)}{3}.$$

Substitutions in Triple Integrals

Substitutions in Triple Integrals

Definition

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ is

$$J(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

The Jacobian determinant can also be denoted by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

Substitutions in Triple Integrals

Theorem (Substitutions in Triple Integrals)

Suppose that $f(x, y, z)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ which is assumed to be one-to-one on the interior of G. If the functions x, y, and z have continuous first partial derivatives within the interior of G, then

$$\iiint_R f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_G f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

Substitutions in Triple Integrals

The Jacobian is simply the scalar triple product of the images of the tangent vectors $\langle \Delta u, 0, 0 \rangle$, $\langle 0, \Delta v, 0 \rangle$, $\langle 0, 0, \Delta w \rangle$, which is the signed volume of the parallelepiped spanned by the image vectors. The absolute value of this gives the distortion in the volume by the map taking *uvw*-space to *xyz*-space.

Example

Example 4

Example

Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of the appropriate region in uvw-space.)

Example 4

Solution

The transformation is given in the problem.

The region R we are integrating over is the interior of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Substituting the equations of the transformation, we find the region G is the interior of the sphere

$$u^2 + v^2 + w^2 = 1$$

in uvw-space. We really don't need to sketch a unit sphere.

Example 4

Solution (cont.)

Now, we compute the Jacobian:

$$J(u,v,w) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$= \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc.$$

Example 4

Solution (cont.)

Now, we apply the change of variables formula.

$$\iiint_R dV = \iiint_G abc\, dV$$
$$= abc \times (\text{volume of unit sphere})$$
$$= \frac{4}{3}\pi abc.$$