Triple Integrals

William M. Faucette

University of West Georgia

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Outline

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Integrable Functions of Three Variables

Integrable Functions of Three Variables

Consider a function $f(x, y, z)$ defined on a box region B,

$$B: \quad a \leq x \leq b, \quad c \leq y \leq d, \quad p \leq z \leq q$$

We divide the box B into small subboxes by choosing a partition of the interval $[a, b]$, a partition of the interval $[c, d]$, and a partition of the interval $[p, q]$.

See the sketch on the next slide.

Integrable Functions of Three Variables

Figure: Partition of Region B

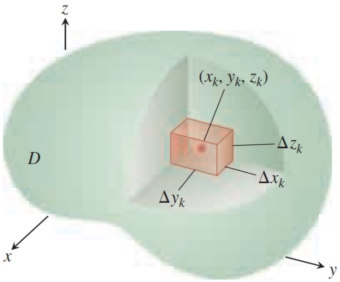


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

Integrable Functions of Three Variables

The subbox has depth Δx_k, width Δy_k, and height Δz_k. So, the volume of the subbox is $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$.

We choose a point (x_k, y_k, z_k) inside the subbox, evaluate the function f at this point, and multiply this by ΔV_k. This gives an approximation of the hypervolume "under" the three-dimensional surface $w = f(x, y, z)$ over the subbox.

We add these up to get a Riemann sum:

$$\sum_k f(x_k, y_k, z_k)\, \Delta V_k$$

Finally, we take the limit as Δx_k, Δy_k, and Δz_k go to zero.

Integrable Functions of Three Variables

Definition

The triple integral of the function $f(x, y, z)$ over a rectangular box B is defined as

$$\iiint_B f(x, y, z)\, dV = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum_k f(x_k, y_k, z_k)\, \Delta V_k,$$

if this limit exists.

Integrable Functions of Three Variables

Let D be a region in the xyz-space and let $f(x, y, z)$ be a positive and continuous function defined on D. Then the integral

$$\iiint_D f(x, y, z)\, dV$$

gives the four-dimensional "hypervolume" "under" the graph $w = f(x, y, z)$ and over the region D.

In order to evaluate this integral, we use Fubini's Theorem again. You just use three nested integrals instead of two.

Integrable Functions of Three Variables

Of course, you don't want to compute the triple integral directly from the definition using Riemann sums, just as you don't want to compute single integrals in Calculus I from the definition using Riemann sums.

Just as for double integrals, we use iterated integrals and Fubini's Theorem, a similar version of which holds for triple integrals.

Fubini's Theorem for Triple Integrals

Theorem 5.9: Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

Setting Limits and Computing

Setting Limits and Computing

Theorem 5.10: Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z) = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA.$$

Setting Limits and Computing

1. Draw the region E of integration.
2. Choose an order of integration. You have six choices. My suggestion is to look at the region of integration, which is a region in space, and project the region into one of the three coordinate planes. You want to project into the plane where you can best see the projection. The image of the projection D will be the region you integrate over for the **outside** two integrals. The remaining variable will be the inside most variable.
3. The outside two limits are set just as you would set a double integral over the **projected** region.

Setting Limits and Computing

4. Lastly, you set the inside limit by fixing a point in the projected region D, looking above it, and figuring out the limits of the line segment inside the 3-dimensional region E.
5. Now evaluate the iterated integral.

Finding a Volume by Evaluating a Triple Integral

Finding a Volume by Evaluating a Triple Integral

To find the volume of a region E in space, you simply evaluate the integral

$$\iiint_E 1\, dV.$$

Examples

Example 1

Example

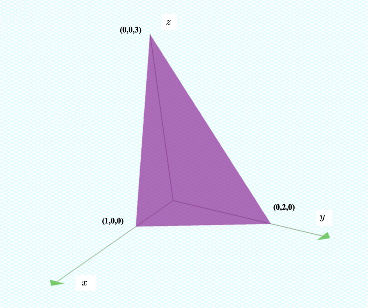
Write one of the six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate the integral.

Example 1

Solution

The first thing you want to do is to sketch the region.

Figure: Sketch of Region E



Example 1

Solution (cont.)

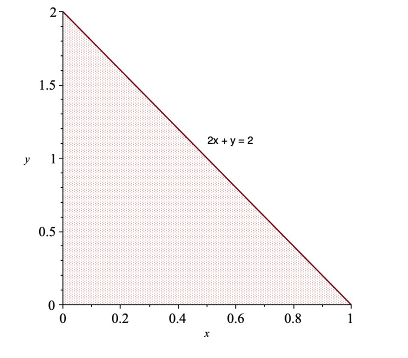
If we project this tetrahedron into the xy-plane, you get the region D. The line in the xy-plane is where the plane $6x + 3y + 2z = 6$ meets the xy-plane, which is given by $z = 0$. Setting $z = 0$ and simplifying, we get $6x + 3y = 6$ or $2x + y = 2$.

See the sketch of the region D on the next slide.

Example 1

Solution (cont.)

Figure: Sketch of Projection D of Region E



Example 1

Solution (cont.)

The volume of the tetrahedron is given by the triple integral

$$V = \iiint_D dV$$

We evaluate this using the figure on the last slide to set the outside two integrals. To trace out the region D, x must go from 0 to 1. For a fixed value of x, y must go from 0 up to the line, where $y = 2 - 2x$.

For fixed values of x and y, i.e. a fixed point in the region D, we look at Figure 2 to see that z must go from 0 up to the plane on the top of the tetrahedron, where $z = 3 - 3x - \frac{3}{2}y$.

Example 1

Solution (cont.)

Now, we compute

$$\begin{aligned}
V = \iiint_D dV &= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} dz\,dy\,dx \\
&= \int_0^1 \int_0^{2-2x} 3 - 3x - \frac{3}{2}y\,dy\,dx \\
&= \int_0^1 3y - 3xy - \frac{3}{4}y^2 \Big|_0^{2-2x} dx \\
&= \int_0^1 \left(3(2-2x) - 3x(2-2x) - \frac{3}{4}(2-2x)^2 \right) dx \\
&= \int_0^1 3(x-1)^2\,dx = (x-1)^3\Big|_0^1 = 1.
\end{aligned}$$

Example

Example

The volume of a solid E is given by the integral

$$\int_{-2}^{0} \int_{x}^{0} \int_{0}^{x^2+y^2} dz\, dy\, dx.$$

Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

Example 2

Solution

We begin by graphing the region E using Mathematica.

The command is

```
ContourPlot3D[{x == -2, y == x, y == 0, z == x^2 + y^2}, {x, -2, 0}, {y, -2, 0}, {z, 0, 4}]
```

Figure: Mathematica Command

Remember you have to press "shift-enter" to execute a command in Mathematica.

Example 2

Solution (cont.)

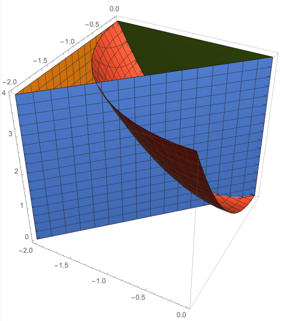


Figure: Graph of Region E

Example 2

Solution (cont.)

Now we use Mathematica to evaluate the iterated triple integral.

```
In[19]:= Integrate[1, {z, 0, x^2 + y^2}]
Out[19]= x^2 + y^2
In[20]:= Integrate[%, {y, x, 0}]
Out[20]= -4 x^3/3
In[21]:= Integrate[%, {x, -2, 0}]
Out[21]= 16/3
```

The answer to two decimal places is 5.33.

Changing the Order of Integration

Changing the Order of Integration

With a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Examples

Example 4

Example

Evaluate the triple integral

$$\iiint_E \sqrt{x^2 + z^2}\, dV,$$

where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Example 4

Solution

First, we sketch the region.

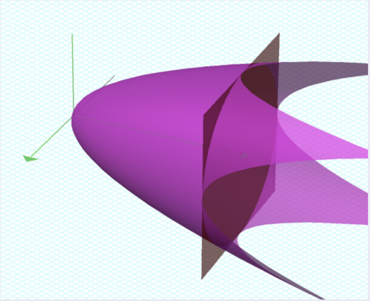


Figure: Sketch of the Region E

Example 4

Solution (cont.)

If we project the region E into the plane $y = 4$, we see we get the disk $x^2 + z^2 \leq 4$. This is a circle of radius 2. We will use this as D, our plane region of integration. For a fixed value of (x, z) in the disk, y goes from $x^2 + z^2$ to 4.

We will use polar coordinates in the xz-coordinates to evaluate the integral.

Example 4

Solution (cont.)

This gives us the integral

$$\iint_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dA.$$

We convert to polar coordinates in the xz-plane to get

$$\int_0^{2\pi} \int_0^{2} \int_{r^2}^{4} r\, dy\, r\, dr\, d\theta$$

Example 4

Solution (cont.)

Now, we compute:

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\, dy\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[ry \right|_{r^2}^4 \, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4r^2 - r^4)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{4}{3} r^3 - \frac{1}{5} r^5 \right|_0^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}.$$

Example 5

Example

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above the sphere $z = x^2 + y^2 + z^2$. Set up a triple integral in spherical coordinates to find the volume of the region, using the order of integration $d\rho\, d\varphi\, d\theta$. Do not evaluate the integral.

Example 5

Solution

First, we sketch the region in space:

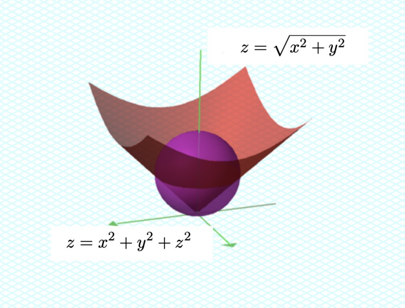


Figure: Sketch for Example 5

Example 5

Solution (cont.)

Notice the region is **above** by the cone $z = \sqrt{x^2 + y^2}$ and **below** the sphere $z = x^2 + y^2 + z^2$. This is the ice cream cone.

This would be the region where φ goes from 0 to $\pi/4$ and ρ goes from 0 to the sphere. The variable θ goes from 0 to 2π (or you could go from 0 to $\pi/2$ and multiply by 4 since the region is symmetric about the z-axis).

Example 5

Solution (cont.)

When a point is on the sphere, we have

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos\varphi \\ \rho &= \cos\varphi \end{aligned}$$

So, ρ goes from 0 to $\cos\varphi$ for a fixed value of φ.

Example 5

Solution (cont.)

We set up the integral and compute:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\rho^3 \sin\varphi \Big|_0^{\cos\varphi} \, d\varphi \, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos^3\varphi \sin\varphi \, d\varphi \, d\theta$$

Example 5

Solution (cont.)

Continuing ...

$$\int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} \cos^3 \varphi \sin \varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3 \varphi \sin \varphi \, d\varphi \, d\theta = \frac{1}{3} \int_0^{2\pi} -\frac{1}{4} \cos^4 \varphi \Big|_0^{\pi/4} \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left(-\frac{1}{4} \cos^4 \left(\frac{\pi}{4} \right) + \frac{1}{4} \cos^4 0 \right) d\theta = \frac{1}{3} \int_0^{2\pi} \frac{3}{16} \, d\theta.$$

Example 5

Solution (cont.)

Finishing . . .

$$\begin{aligned}\frac{1}{3}\int_0^{2\pi}\frac{3}{16}\,d\theta &= \frac{1}{16}\left[\theta\right|_0^{2\pi} \\ &= \frac{1}{16}\cdot 2\pi \\ &= \frac{\pi}{8}.\end{aligned}$$

Average Value of a Function of Three Variables

Average Value of a Function of Three Variables

Definition: Average Value of a Function of Three Variables

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume then the average value of the function is

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

Here, $V(E)$ is the volume of the region E.

Example

Example 3

Example

Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped E determined by $x = 0$, $x = 1$, $y = 0$, $y = 3$, $z = 0$, $z = 5$.

Average Value of a Function of Three Variables

Solution

We first compute the volume of the region:

$$\begin{aligned}\iiint_E dV &= \int_0^1 \int_0^3 \int_0^5 dz\, dy\, dx \\ &= \int_0^1 \int_0^3 5\, dy\, dx \\ &= \int_0^1 15\, dx \\ &= 15.\end{aligned}$$

Of course, this is just the volume of the box E.

Average Value of a Function of Three Variables

Solution (cont.)

Next, we compute the integral of the function over the region:

$$\begin{aligned}
\iiint_E dV &= \int_0^1 \int_0^3 \int_0^5 x + y + z \, dz \, dy \, dx \\
&= \int_0^1 \int_0^3 \left[xz + yz + \frac{1}{2} z^2 \right|_0^5 dy \, dx \\
&= \int_0^1 \int_0^3 \left(5x + 5y + \frac{25}{2} \right) dy \, dx
\end{aligned}$$

Average Value of a Function of Three Variables

Solution (cont.)

Continuing,

$$\iiint_E dV = \int_0^1 \int_0^3 \left(5x + 5y + \frac{25}{2}\right)\, dy\, dx$$
$$= \int_0^1 \left[5xy + \frac{5}{2}y^2 + \frac{25}{2}y\right|_0^3 \, dx$$
$$= \int_0^1 (15x + 60)\, dx$$
$$= \frac{15}{2}x^2 + 60x\Big|_0^1 = \frac{135}{2}.$$

Average Value of a Function of Three Variables

Solution (cont.)

So, the average value of the function is

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV = \frac{1}{15} \cdot \frac{135}{2} = \frac{9}{2}.$$