Directional Derivatives and the Gradient

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Outline

Directional Derivatives in the Plane

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Suppose we have a differentiable function $f(x, y)$ and a differentiable curve $x = x(t)$, $y = y(t)$. By the Chain Rule, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Directional Derivatives in the Plane

Now suppose $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is a unit vector and (x_0, y_0) is a point in the domain of f. The line in the xy-plane through (x_0, y_0) in the direction $\mathbf{u}$ has parametric equations

$$\begin{cases} x = x_0 + s\cos\theta \\ y = y_0 + s\sin\theta. \end{cases}$$

Directional Derivatives in the Plane

On the surface $z = f(x, y)$ above this line we get a curve

$$\begin{cases} x = x_0 + s\cos\theta \\ y = y_0 + s\sin\theta \\ z = f(x_0 + s\cos\theta, y_0 + s\sin\theta). \end{cases}$$

passing through $(x_0, y_0, f(x_0, y_0))$.

The derivative $df/ds(0)$ is the slope of this curve at $(x_0, y_0, f(x_0, y_0))$. This is the slope of the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ in the direction $\mathbf{u}$.

Directional Derivatives in the Plane

Definition

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Then the **directional derivative** of f in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h},$$

provided the limits exists.

In this definition, $\mathbf{u}$ <u>must</u> be a unit vector.

Directional Derivatives in the Plane

Notation

The **directional derivative** just defined is denoted by

$$D_{\mathbf{u}}f(P_0), \quad D_{\mathbf{u}}f|_{P_0}, \quad \left(\frac{df}{ds}\right)_{\mathbf{u},P_0}.$$

Example

Example 1

Example

Find the derivative of the function $f(x, y) = 2x^2 + y^2$ at the point $P_0(-1, 1)$ in the direction of $\mathbf{u} = \frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}$.

Example 1

Solution

$$\begin{aligned}
D_{\mathbf{u}}f(P_0) &= \lim_{s\to 0}\frac{f\left(-1+\frac{3}{5}s, 1-\frac{4}{5}s\right)-f(-1,1)}{s}\\
&= \lim_{s\to 0}\frac{2\left(-1+\frac{3}{5}s\right)^2+\left(1-\frac{4}{5}s\right)^2-3}{s}\\
&= \lim_{s\to 0}\frac{\frac{34}{25}s^2-4s}{s}\\
&= \lim_{s\to 0}\frac{34}{25}s-4\\
&= -4.
\end{aligned}$$

Interpretation of the Directional Derivative

Interpretation of the Directional Derivative

The directional derivative takes the vertical plane through $(x_0, y_0, f(x_0, y_0))$ defined by

$$\begin{cases} x = x_0 + tu_1 \\ y = y_0 + tu_2, \end{cases}$$

which cuts out on the the surface $z = f(x, y)$ a curve

$$\begin{cases} x = x_0 + tu_1 \\ y = y_0 + tu_2 \\ z = f(x_0 + tu_1, y_0 + tu_2). \end{cases}$$

Interpretation of the Directional Derivative

The directional derivative is the rate change of the function $f(x, y)$ along this curve lying on the surface $z = f(x, y)$.

The directional derivative of f at P_0 in the direction $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is the slope of this curve at P_0.

The directional derivative tells you the rate of change of the surface $z = f(x, y)$ at the point P_0 in the direction $\mathbf{u}$.

Calculation and Gradients

Calculation and Gradients

Using the Chain Rule, the directional derivative of f at P_0 in the direction $\mathbf{u}$ can be written in the form

$$\begin{aligned}\left(\frac{df}{dt}\right)_{\mathbf{u},P_0} &= \frac{\partial f}{\partial x}(P_0)\frac{dx}{dt} + \frac{\partial f}{\partial y}(P_0)\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}(P_0)\cos\theta + \frac{\partial f}{\partial y}(P_0)\sin\theta\\ &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)(P_0)\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\\ &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)(P_0)\cdot\mathbf{u}.\end{aligned}$$

Calculation and Gradients

Definition

The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}.$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f|_{P_0} \quad \text{or} \quad \nabla f(P_0) \quad \text{or} \quad \nabla f(x_0, y_0).$$

The symbol ∇f is read "grad f".

Calculation and Gradients

With this new notation, our previous calculation gives us the following theorem.

The Directional Derivative Is a Dot Product

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$, then

$$D_{\mathbf{u}}f(P_0) = \nabla f|_{P_0} \cdot \mathbf{u},$$

the dot product of the gradient ∇f at P_0 with the vector $\mathbf{u}$. In brief $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

Example

Example 2

Example

We look at Example 1 again. Find the derivative of the function $f(x, y) = 2x^2 + y^2$ at the point $P_0(-1, 1)$ in the direction of $\mathbf{u} = \frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}$.

Example 2

Solution

We compute

$$\begin{aligned}
\nabla f &= 4x\,\mathbf{i} + 2y\,\mathbf{j} \\
\nabla f(P_0) &= 4(-1)\,\mathbf{i} + 2(1)\,\mathbf{j} = -4\,\mathbf{i} + 2\,\mathbf{j} \\
D_{\mathbf{u}} f(P_0) &= (-4\,\mathbf{i} + 2\,\mathbf{j}) \cdot \left(\frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}\right) \\
&= (-4)\,\frac{3}{5} + 2\left(-\frac{4}{5}\right) = \frac{-20}{5} = -4,
\end{aligned}$$

just as before.

Properties of the Directional Derivative

Properties of the Directional Derivative

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is in the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}} f = \|\nabla f\|.$$

2. Similarly, f decreasing most rapidly in the direction of $-\nabla f$. The derivative in this direction of $D_{\mathbf{u}} f = -\|\nabla f\|$.
3. Any direction $\mathbf{u}$ is orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because of θ then equals $\pi/2$ and

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\left(\frac{\pi}{2}\right) = \|\nabla f\| \cdot 0 = 0.$$

Example

Example 3

Example

Find the directions in which $f(x, y) = x^2 y + e^{xy} \sin y$ …

a …increases most rapidly at the point $(1, 0)$ …

b …decreases most rapidly at the point $(1, 0)$ …

at the point $(1, 0, 0)$.

Example 3

Solution

- **a** The function increases most rapidly in the direction of the gradient at this point. We compute

  $$\nabla f = (2xy + ye^{xy}\sin y)\,\mathbf{i} + (x^2 + xe^{xy}\sin y + e^{xy}\cos y)\,\mathbf{j}$$

  $$\nabla f(1,0) = 2\,\mathbf{j}$$

  The maximum increase occurs in the direction $\mathbf{j}$.
- **b** The maximum decrease occurs in the opposite direction, $-\mathbf{j}$.

Gradients and Tangents to Level Curves

Gradients and Tangents to Level Curves

Suppose the function $f(x, y)$ has the constant value c along a curve $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. That is, the curve $\mathbf{r}$ is a parametrized level curve with tangent vector $\mathbf{r}'$.

Then we have

$$\begin{aligned}
\frac{d}{dt} f(x(t), y(y)) &= \frac{d}{dt} c \\
\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= 0 \\
\left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}\right) \cdot \left(\frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}\right) &= 0 \\
\nabla f \cdot \mathbf{r}' &= 0.
\end{aligned}$$

Gradients and Tangents to Level Curves

Suppose the function $f(x, y)$ has the constant value c along a curve $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$.

Since

$$\nabla f \cdot \mathbf{r}' = 0$$

for all values of t, we have the following result:

Gradients are Orthogonal to Level Curves

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Gradients and Tangents to Level Curves

The tangent line to a curve at a point is perpendicular to the normal vector to the level curve, which is the gradient of f at the point of tangency. So, the equation of the tangent line to the curve at the point (x_0, y_0) is

$$\nabla f(P_0) \cdot ((x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j}) = 0$$

Gradients and Tangents to Level Curves

This yields the following result:

Tangent Line to a Level Curve

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

Example

Example 4

Example

Find the equation for the tangent line to the curve $x^2 - y = 1$ at the point $(\sqrt{2}, 1)$.

Example 4

Solution

If $f(x, y) = x^2 - y$, then this line is tangent to a level curve of the surface $z = f(x, y)$. We compute

$$\begin{array}{ll} f_x(x, y) = 2x & f_y(x, y) = -1 \\ f_x(\sqrt{2}, 1) = 2\sqrt{2} & f_y(\sqrt{2}, 1) = -1\ . \end{array}$$

The tangent line is then

$$2\sqrt{2}(x - \sqrt{2}) - (y - 1) = 0.$$

Differentiation Rules for Gradients

Differentiation Rules for Gradients

Differentiation Rules for Gradients

Sum Rule:	$\nabla(f+g)=\nabla f+\nabla g$
Difference Rule:	$\nabla(f-g)=\nabla f-\nabla g$
Constant Multiple Rule:	$\nabla(kf)=k\nabla f$ any number k
Product Rule:	$\nabla(fg)=f\nabla g+g\nabla f$
Quotient Rule:	$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$

Functions of Three Variables

Functions of Three Variables

For a differentaible function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ we have

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

and

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3. \end{aligned}$$

We still have that the maximum rate of change occurs in the direction of the gradient and the value of that maximum change is the length of the gradient vector.

Example

Example 5

Example

- **a** Find the derivative of $f(x, y, z) = xe^y + z^2$ at $P_0(1, \ln 2, 1/2)$ in the direction $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$.
- **b** In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Example 5

Solution

We compute

$$\begin{aligned}
\nabla f &= e^{y}\,\mathbf{i} + xe^{y}\,\mathbf{j} + 2z\,\mathbf{k} \\
\nabla f(P_0) &= e^{\ln 2}\,\mathbf{i} + (1)e^{\ln 2}\,\mathbf{j} + 2(1/2)\,\mathbf{k} \\
&= 2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}.
\end{aligned}$$

Example 5

Solution (cont.)

We normalize $\mathbf{v}$ to get the unit vector $\mathbf{u} = \frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$.

a. $Df_{\mathbf{u}}(P_0) = (2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}) \cdot (\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}) = \frac{8}{3}$.

b. The function increases most rapidly in the direction of the gradient: $2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}$. The value of this increase is the length of the gradient vector: 3.

   The function decreases most rapidly in the opposite direction: $-2\,\mathbf{i} - 2\,\mathbf{j} - 1\,\mathbf{k}$. The value of this decrease is minus the length of the gradient vector: -3.

The Chain Rule for Paths

The Chain Rule for Paths

If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is a smooth path C, and $w = f(\mathbf{r}(t))$ is a scalar function evaluated along C, the by the Chain Rule we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\\ &= \left(\frac{\partial w}{\partial x}\,\mathbf{i} + \frac{\partial w}{\partial y}\,\mathbf{j} + \frac{\partial w}{\partial z}\,\mathbf{k}\right)\cdot\left(\frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\right)\\ &= \nabla f \cdot \mathbf{r}',\end{aligned}$$

where the partial derivatives are evaluated at a point on the curve.

The Chain Rule for Paths

This gives us the following result:

The Derivative Along a Path

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$