The Chain Rule

William M. Faucette

University of West Georgia

Fall 2025

Outline

Functions of Two Variables

Functions of Two Variables

The Chain Rule for functions of one variable says that if w is a differentiable function of x and x is a differentiable function of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

We now develop the Chain Rule for functions of more than one variable.

Functions of Two Variables

Suppose $z = f(x, y)$ is differentiable and $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then we have the following theorem.

Theorem

If $z = f(x, y)$ is differentiable and $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composition $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Examples

Example 1

Example

Let $w = x^2 + y^2$, $x = \cos t + \sin t$, and $y = \cos t - \sin t$. Express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t.

Example 1

Solution

First, we use the Chain Rule.

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= 2x \cdot (\cos t - \sin t) + 2y \cdot (-\cos t - \sin t) \\
&= 2(\cos t + \sin t)(\cos t - \sin t) \\
&\qquad + 2(\cos t - \sin t)(-\cos t - \sin t) \\
&= 2(\cos t + \sin t)(\cos t - \sin t) \\
&\qquad - 2(\cos t - \sin t)(\cos t + \sin t) \\
&= 0.
\end{aligned}$$

From this, we see that $dw/dt = 0$.

Solution (cont.)

Now we substitute the expressions for x and y into w and then take the derivative.

$$\begin{aligned} w &= x^2 + y^2 \\ &= (\cos t + \sin t)^2 + (\cos t - \sin t)^2 \\ &= (\cos^2 t + 2\cos t \sin t + \sin^2 t) \\ &\qquad\qquad + (\cos^2 t - 2\cos t \sin t + \sin^2 t) \\ &= 2\cos^2 t + 2\sin^2 t = 2(\cos^2 t + \sin^2 t) = 2. \end{aligned}$$

From this, we also see that $dw/dt = 0$.

Functions of Three Variables

Suppose $w = f(x, y, z)$ is differentiable and $x = x(t)$, $y = y(t)$, and $z = z(t)$ are differentiable functions of t, then we have the following theorem.

Theorem

If $w = f(x, y, z)$ is differentiable and $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable functions of t, then the composition $w = f(x(t), y(t), z(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example 2

Example

Let $w = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, and $z = 4\sqrt{t}$. Express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t.

Example 2

Solution

First, we use the Chain Rule.

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t)\\ &\qquad + \frac{2z}{x^2+y^2+z^2}(2t^{-1/2})\end{aligned}$$

We compute that $x^2 + y^2 + z^2 = 1 + 16t$.

Example 2

Solution (cont.)

Substituting $x^2 + y^2 + z^2 = 1 + 16t$, we get

$$\begin{aligned}\frac{dw}{dt} &= \frac{2\cos t}{1+16t}(-\sin t) + \frac{2\sin t}{1+16t}(\cos t) + \frac{2\cdot 4\sqrt{t}}{1+16t}(2t^{-1/2}) \\ &= \frac{16}{1+16t},\end{aligned}$$

since the first two terms cancel and in the third term, $\sqrt{t}$ and $t^{-1/2}$ cancel.

Example 2

Solution (cont.)

Now we substitute the expressions for x, y, and z into w and then take the derivative.

$$\begin{aligned} w &= \ln(x^2 + y^2 + z^2) \\ &= \ln((\cos t)^2 + (\sin t)^2 + (4\sqrt{t})^2) \\ &= \ln(1 + 16t). \end{aligned}$$

From this, we see that $dw/dt = 16/(1 + 16t)$.

Chain Rule for Two Independent Variables

Chain Rule for Two Independent Variables

Theorem 4.9: Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$, $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then z is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

The Generalized Chain Rule

The Generalized Chain Rule

Theorem 4.10: Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each i, $1 \le i \le m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any j, $1 \le j \le n$.

Implicit Differentiation

Implicit Differentiation

Suppose the function $z(x, y)$ is differentiable and the equation $z(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = f(x)$.

Since $w = z(x, y) = 0$ on the curve $y = f(x)$, we find

$$\begin{aligned} 0 = \frac{dw}{dx} &= \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \end{aligned}$$

So,

$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y} = -\frac{z_x}{z_y}.$$

Implicit Differentiation

This gives us the following theorem:

Theorem 4.11: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z = f(x, y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x, y) = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

provide $f_y(x, y) \neq 0$.

Implicit Differentiation

This gives us the following theorem if f is a function of three variables:

Theorem 4.11: Implicit Differentiation of a Function of Two or More Variables

If the equation $f(x, y, z) = 0$ defines z implicitly as a function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \text{ and } \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

provide $f_z(x, y, z) \neq 0$.

Examples

Example 5

Example

Use implicit differentiation to find dy/dx if $y^2 - x^2 - sin(xy) = 0$.

Example 5

Solution

Here we have $F(x, y) = y^2 - x^2 - sin(xy)$. From the last theorem, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos(xy)}{2y - x\cos(xy)} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

Implicit Differentiation

Suppose the function $F(x, y, z)$ is differentiable and the equation $w = F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, say $z = f(x, y)$.

Implicit Differentiation

Since $F(x, y, z) = 0$ on the curve $z = f(x, y)$, we find

$$\begin{aligned} 0 = \frac{\partial w}{\partial x} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial y} \cdot 0 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}. \end{aligned}$$

So,

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z}.$$

Implicit Differentiation

An important result from advanced calculus, called the **Implicit Function Theorem**, states the conditions for which our results

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

are valid. If the partial derivatives F_x, F_y, and F_z are continuous throughout an open region $\mathcal{R}$ in space containing the point (x_0, y_0, z_0), and if for some contstant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by equations above.

Example 6

Example

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$, assuming that $F_z \neq 0$.

Example 6

Solution

Here we have $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. From the last theorem, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + yze^{xz}}{2z + xye^{xz} + \cos y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

Functions of Many Variables

In general, suppose that $w = f(x, y, \ldots, v)$ is a differentiable function of the intermediate variables $x, y, \ldots, v$ (a finite set) and the $x, y, \ldots, v$ are differentiable functions of the independent variables p, q, . . . , t (another finite set). Then w is a differentiable function of the variables p through t, and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

The other equations are obtained by replacing p by $q, \ldots, t$, one at a time.

Functions of Many Variables

One way to remember this equation is to think of the right-hand side as the dot product of two vectors with components

$$\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots, \frac{\partial w}{\partial v}\right) \quad \text{and} \quad \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \dots, \frac{\partial v}{\partial p}\right).$$