

# Tangent Planes and Linear Approximations

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# Tangent Planes and Normal Lines

# Tangent Planes and Normal Lines

Suppose  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  is a smooth curve on a level surface  $f(x, y, z) = c$ . In the last section, we found that the gradient of  $f$  is normal to  $\mathbf{r}'(t)$ .

Suppose  $P_0 = (x_0, y_0, z_0)$  is a point on the curve  $\mathbf{r}(t)$ . Then we have a point on the surface and a normal vector. So we can write the equation of the **tangent plane**.

$$((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) \cdot \nabla f(P_0) = 0.$$

or

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

# Tangent Planes and Normal Lines

Notice that the equation of the tangent plane simplifies to

$$f_x(P_0)x + f_y(P_0)y + f_z(P_0)z = D,$$

where  $D$  is the value of the expression on the left evaluated at  $P_0$ .

# Tangent Planes and Normal Lines

## Definition

The **tangent plane** to the level surface  $f(x, y, z) = c$  of a differentiable function  $f$  at a point  $P_0$  where the gradient is not zero is the plane through  $P_0$  normal to  $\nabla f|_{P_0}$ .

The **normal line** of the surface at  $P_0$  is the line through  $P_0$  parallel to  $\nabla f|_{P_0}$ .

# Tangent Planes and Normal Lines

**Tangent Plane to  $f(x, y, z) = c$  at  $P_0(x_0, y_0, z_0)$ :**

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

or

$$f_x(P_0)x + f_y(P_0)y + f_z(P_0)z = D$$

where,  $D = f_x(P_0)x_0 + f_y(P_0)y_0 + f_z(P_0)z_0$ .

**Normal Line to  $f(x, y, z) = c$  at  $P_0(x_0, y_0, z_0)$ :**

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

## Example

# Example 1

## Example

Find equations for the tangent plane and normal line at the point  $P_0(3, 5, -4)$  on the surface  $x^2 + y^2 - z^2 = 18$ .

# Example 1

## Solution

We compute

$$\begin{aligned}\nabla f &= 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \\ \nabla f(P_0) &= 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}\end{aligned}$$

The equation of tangent plane is then

$$6(x - 3) + 10(y - 5) + 8(z + 4) = 0$$

and the normal line is

$$x = 3 + 6t, \quad y = 5 + 10t, \quad z = -4 + 8t.$$

Plane Tangent to a Surface  $z = f(x, y)$

## Plane Tangent to a Surface $z = f(x, y)$

Let  $z = f(x, y)$  be a smooth surface and let  $P_0 = (x_0, y_0, z_0)$  be a point on the surface.

If we look in the  $x$ -direction, the slope of the surface at  $P_0$  is  $f_x(x_0, y_0)$ , so one tangent vector to the surface at  $P_0$  is  $\langle 1, 0, f_x(x_0, y_0) \rangle$ .

Similarly, if we look in the  $y$ -direction, the tangent vector to the surface at  $P_0$  is  $\langle 0, 1, f_y(x_0, y_0) \rangle$ .

## Plane Tangent to a Surface $z = f(x, y)$

A normal vector to the surface at the point  $P_0$  is the cross product of these two vectors:  $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$ .

So, the equation of the tangent plane to  $z = f(x, y)$  at the point  $P_0 = (x_0, y_0, z_0)$  is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

# Plane Tangent to a Surface $z = f(x, y)$

## Tangent Plane to a Surface at a Point

The plane tangent to the surface  $z = f(x, y)$  of a differentiable function  $f$  at the point  $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$  is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

## Examples

## Example 2

### Example

Find an equation for the plane that is tangent to the surface  $z = \ln(x^2 + y^2)$  at the point  $(1, 0, 0)$ .

## Example 2

### Solution

We compute using the formula

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

getting

$$\frac{2x}{x^2 + y^2} \Big|_{(1,0)} (x - 1) + \frac{2y}{x^2 + y^2} \Big|_{(1,0)} (y - 0) - (z - 0) = 0$$
$$2(x - 1) - z = 0.$$

## Example 3

### Example

Find parametric equations for the line tangent to the curve of intersection of the surfaces  $x + y^2 + 2z = 4$  and  $x = 1$  at the point  $(1, 1, 1)$

## Example 3

### Solution

Both these are level surfaces for the functions  $g(x, y, z) = x + y^2 + 2z$  and  $h(x, y, z) = x$ , so a normal vector to each surface is the gradient at the point  $(1, 1, 1)$ .

This gives us two vectors normal to the surface at the point  $(1, 1, 1)$ .

Their cross product will then be the direction vector  $\mathbf{v}$  of the tangent line.

## Example 3

### Solution (cont.)

We compute

$$\nabla g = \langle 1, 2y, 2 \rangle, \quad \nabla g(1, 1) = \langle 1, 2, 2 \rangle$$

$$\nabla h = \langle 1, 0, 0 \rangle, \quad \nabla h(1, 1) = \langle 1, 0, 0 \rangle$$

$$\mathbf{v} = \langle 1, 2, 2 \rangle \times \langle 1, 0, 0 \rangle = \langle 0, 2, -2 \rangle$$

So, the equations of the tangent line to the curve are

$$\begin{cases} x = 1 \\ y = 1 + 2t \\ z = 1 - 2t. \end{cases}$$

# Linear Approximations

# Linear Approximations

In Calculus 1 you learned how to perform linear approximations of differentiable functions using tangent lines.

The linear approximation of a function  $f(x)$  at the point  $x = a$  is given by

$$y \approx f(a) + f'(a)(x - a).$$

# Linear Approximations

Similarly we can approximate the value of a function  $f(x, y)$  near a point  $(x_0, y_0)$  on the graph  $z = f(x, y)$  using the tangent plane to the surface at the point. Recall, the tangent plane has equation

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

If we solve this for  $z$ , we get

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

# Linear Approximations

## Definition

The **linearization** of a function  $f(x, y)$  at a point  $(x_0, y_0)$  where  $f$  is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of  $f$  at  $(x_0, y_0)$ .

## Example 5

### Example

Find the linearization  $L(x, y)$  of the function

$$f(x, y) = (x + y + 2)^2$$

at the point  $(1, 2)$ .

## Example 5

### Solution

We compute

$$f(1, 2) = (1 + 2 + 2)^2 = 25$$

$$f_x(x, y) = f_y(x, y) = 2(x + y + 2)$$

$$f_x(1, 2) = f_y(1, 2) = 2(1 + 2 + 2) = 10.$$

The linearization is then

$$L(x, y) = 25 + 10(x - 1) + 10(y - 2) = 10x + 10y - 5.$$

# Differentiability

The concept of differentiability for functions of several variables is more complicated than for single-variable functions because a point in the domain can be approached along more than one path.

We start by reframing the definition of differentiability from Calculus 1.

# Differentiability

Suppose  $y = f(x)$  is differentiable at  $x_0$ . Then

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Let

$$\varepsilon = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0).$$

Then

$$f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \varepsilon\Delta x$$

where  $\varepsilon \rightarrow 0$  as  $\Delta x \rightarrow 0$ .

This says the difference in the function  $f(x_0 + \Delta x) - f(x_0)$  is well-approximated by the difference in the tangent line:  $f'(x_0)\Delta x$ .

# Differentiability

## Definition

A function  $z = f(x, y)$  is **differentiable at**  $(x_0, y_0)$  if  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  and  $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$  satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of  $\varepsilon_1, \varepsilon_2 \rightarrow 0$  as both  $\Delta x, \Delta y \rightarrow 0$ . We call  $f$  **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

This says the difference in the function  $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$  is well-approximated by the difference in the tangent plane:  $f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y$ .

## The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of  $f(x, y)$  are defined throughout an open region  $R$  containing the point  $(x_0, y_0)$  and that  $f_x$  and  $f_y$  are continuous at  $(x_0, y_0)$ . Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of  $f$  that results from moving from  $(x_0, y_0)$  to another point  $(x_0 + \Delta x, y_0 + \Delta y)$  in  $R$  satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of  $\varepsilon_1, \varepsilon_2 \rightarrow 0$  as both  $\Delta x, \Delta y \rightarrow 0$ .

# Differentiability

It is not enough for  $f_x(x_0, y_0)$ ,  $f_y(x_0, y_0)$  to exist for  $f$  to be differentiable. However, if  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  exist in an open disk containing  $(x_0, y_0)$  and are continuous at  $(x_0, y_0)$ , then  $f$  is differentiable. This is the content of the last slide.

## Corollary

If the partial derivative  $f_x$  and  $f_y$  of a function  $f(x, y)$  are continuous throughout an open region  $R$ , then  $f$  is differentiable at every point of  $R$ .

We also have this result analogous to the similar result in Calculus 1. If  $f$  is differentiable at a point, then  $f$  is continuous at that point.

## Differentiability Implies Continuity

If  $f(x, y)$  is differentiable at  $(x_0, y_0)$ , then  $f$  is continuous at  $(x_0, y_0)$ ,

# Differentials

# Differentials

Let  $f(x, y)$  be a differentiable function near a point  $(x_0, y_0)$ . Let  $\Delta f = f(x, y) - f(x_0, y_0)$  be the change in  $f$  as the input changes from  $(x_0, y_0)$  to a nearby point  $(x_0 + \Delta x, y_0 + \Delta y)$ .

This change can be approximated by the change on the tangent plane between these two points.

This change is

$$\begin{aligned}\Delta L &= L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0) \\ &= f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y.\end{aligned}$$

So

$$\Delta f \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y.$$

## Definition

If we move from  $(x_0, y_0)$  to a point  $(x_0 + dx, y_0 + dy)$  nearby, the resulting change

$$df = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$$

in the linearization of  $f$  is called the **total differential of  $f$** .

## Example 6

### Example

The function  $W(v, T)$  is defined by

$$W(v, T) = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

Use the value of  $W(25, 5) = -17.4088$  and differentials to approximate  $W(24, 6)$ .

## Example 6

### Solution

We use the point  $(25, 5)$  as the given point and the point  $(24, 6)$  as the nearby point. Then  $dv = \Delta v = -1$  and  $dT = \Delta T = 1$ . We compute the differentials

$$df \approx -0.36 dv + 1.337 dT = -0.36(-1) + 1.337(1) = 1.697.$$

So

$$W(24, 6) \approx W(25, 5) + df = -17.4088 + 1.697 = -15.7118.$$

# Functions of More Than Two Variables

# Functions of More Than Two Variables

Analogous results hold for differentiable functions of more than two variables.

- 1 The **linearization** of  $f(x, y, z)$  at point  $P_0(x_0, y_0, z_0)$  is

$$\begin{aligned}L(x, y, z) = & f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) \\ & + f_y(x_0, y_0, z_0)(y - y_0) \\ & + f_z(x_0, y_0, z_0)(z - z_0).\end{aligned}$$

- 2 If the second partial derivatives of  $f$  are continuous and if  $x$ ,  $y$ , and  $z$  change from  $x_0$ ,  $y_0$ , and  $z_0$  by small amounts  $dx$ ,  $dy$ , and  $dz$ , the **total differential**

$$df = f_x(x_0, y_0, z_0) dx + f_y(x_0, y_0, z_0) dy + f_z(x_0, y_0, z_0) dz$$

gives a good approximation of the resulting change in  $f$ .