Functions of Several Variables

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Outline

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Definition

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** $x_1, x_2, \ldots, x_n$.

Notice the domain of a function of n variables is a subset of $\mathbb{R}^n$. In particular, the domain of a function $f(x, y)$ is a region in the plane and the domain of a function $f(x, y, z)$ is a region in space.

Domains and Ranges

Domains and Ranges

In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero.

The domain of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly.

The range consists of the set of output values for the dependent variable.

Example 1

Example 1

Example

Find the domain of the function $f(x, y) = \sqrt{y - x - 2}$.

Example 1

Solution

In order to take the square root and get a real number, we must have $y - x - 2 \geq 0$. So, the domain of this function is the half-plane $y \geq x + 2$.

See the sketch of the domain on the next slide.

Example 1

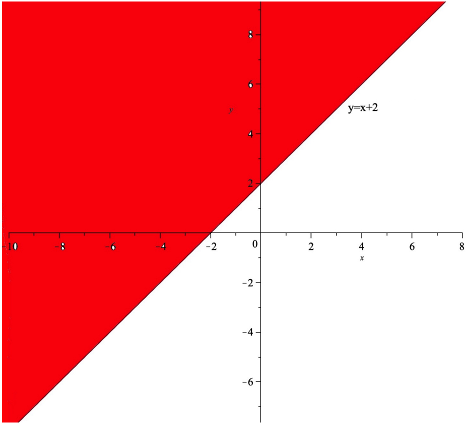


Figure: Domain for Example 1

Functions of Two Variables

Functions of Two Variables

It's time for a bit of terminology from general topology.

Definition

A point (x_0, y_0) in a region R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R.

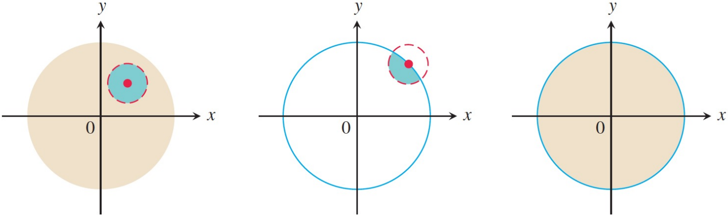
A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**.

A region is **open** if is consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Functions of Two Variables

Figure: Sketches of Interior and Boundary Points



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all
boundary points.

Functions of Two Variables

Definition

A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Example 2

Example 2

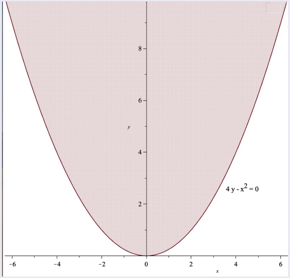
Example

Describe the domain of the function $f(x, y) = \sqrt{4y - x^2}$

Example 2

Solution

Figure: Domain for Example 2



The domain is the inside of this parabola $4y - x^2 = 0$ and the parabola itself. This region is closed and unbounded.

Graphs, Level Curves, and Contours of Functions of Two Variables

Graphs, Level Curves, and Contours of Functions of Two Variables

Definition

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of r.

The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Example 3

Example 3

Example

Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$ $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Example 3

Solution

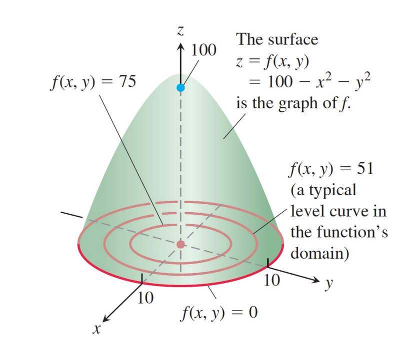


Figure: Sketch for Example 3

Functions of Three Variables

Functions of Three Variables

Definition

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = k$ is called a **level surface** of f.

Example 4

Example 4

Example

Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Example 4

Solution

Figure: Sketch for Example 4