Motion In Space

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Outline

Motion Vectors in the Plane and in Space

Motion Vectors in the Plane and in Space

Definition

Let $\mathbf{r}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time. The **velocity vector** $\mathbf{v}(t)$ of the object is given by

$$\text{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t).$$

The **acceleration vector** $\mathbf{a}(t)$ is defined to be

$$\text{Acceleration} = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

The **speed** is defined to be

$$\text{Speed} = v(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt}.$$

Motion Vectors in the Plane and in Space

Since $\mathbf{r}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ and in three dimensions $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Quantity	**Two Dimensions**	**Three Dimensions**
Position	$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$	$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$
Velocity	$\mathbf{v}(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$	$\mathbf{v}(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}$
Acceleration	$\mathbf{a}(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j}$	$\mathbf{a}(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)\,\mathbf{k}$
Speed	$v(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$	$v(t) = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

Components of the Acceleration Vector

Components of the Acceleration Vector

Theorem 3.7: The Plane of the Acceleration Vector

The acceleration vector $\mathbf{a}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\mathbf{r}(t)$ lies in the plane formed by the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$ to C. Furthermore,

$$\mathbf{a}(t) = \frac{dv}{dt}\,\mathbf{T}(t) + \kappa v^2\,\mathbf{N}(t).$$

Here, $v = v(t)$ is the speed of the object and κ is the curvature of C traced out by $\mathbf{r}(t)$.

Components of the Acceleration Vector

Proof.

Since $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, we have

$$\mathbf{v}(t) = \|\mathbf{r}'(t)\|\,\mathbf{T}(t) = v(t)\,\mathbf{T}(t).$$

Taking the derivative of this equation yields

$$\mathbf{a}(t) = \mathbf{v}'(t) = v'(t)\,\mathbf{T}(t) + v(t)\,\mathbf{T}'(t).$$

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Proof (cont.)

Since $\mathbf{N}(t) = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$, we know that $\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t)$, so

$$\begin{aligned} \mathbf{a}(t) &= v'(t)\,\mathbf{T}(t) + v(t)\,\mathbf{T}'(t) \\ &= v'(t)\,\mathbf{T}(t) + v(t)\,\|\mathbf{T}'(t)\|\,\mathbf{N}(t). \end{aligned}$$

Motion Vectors in the Plane and in Space

Proof (cont.)

One formula for curvature is

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|},$$

so $\|\mathbf{T}'(t)\| = \kappa\|\mathbf{r}'(t)\| = \kappa v(t)$.

Substituting gives us

$$\begin{aligned} \mathbf{a}(t) &= v'(t)\,\mathbf{T}(t) + v(t)\,\|\mathbf{T}'(t)\|\,\mathbf{N}(t) \\ &= v'(t)\,\mathbf{T}(t) + \kappa[v(t)]^2\,\mathbf{N}(t). \end{aligned}$$

This is the desired result. □

Motion Vectors in the Plane and in Space

In the decomposition

$$\mathbf{a}(t) = \frac{dv}{dt}\,\mathbf{T}(t) + \kappa v^2\,\mathbf{N}(t).$$

the coefficient of $\mathbf{T}(t)$ is the **tangential component of acceleration** and the coefficient of $\mathbf{N}(t)$ is the **normal component of acceleration**, respectively, which we write as $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$.

Motion Vectors in the Plane and in Space

Theorem 3.8: Tangential and Normal Components of Acceleration

Let $\mathbf{r}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\mathbf{a}(t) = \mathbf{r}''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ are given by the formulas

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

and

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}.$$

Projectile Motion

Projectile Motion

Newton's second law also tells us that $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_g \\ m\mathbf{a} &= -mg\,\mathbf{j} \\ \mathbf{a} &= -g\,\mathbf{j} \end{aligned}$$

Projectile Motion

To derive the equation of projectile motion, we start with the acceleration due to gravity:

$$\mathbf{a}(t) = -g\,\mathbf{j}.$$

We integrate with respect to t to get

$$\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}.$$

Setting $t = 0$, we get $\mathbf{v}(0) = \mathbf{C}$. Call this vector $\mathbf{v}_0$, the initial velocity. So,

$$\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0.$$

Projectile Motion

We start with the velocity function from the last slide:

$$\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0.$$

We integrate with respect to t to get

$$\mathbf{s}(t) = -\frac{1}{2}gt^2\,\mathbf{j} + \mathbf{v}_0 t + \mathbf{C}.$$

Setting $t = 0$, we get $\mathbf{s}(0) = \mathbf{C}$, which is the initial position. We'll denote this $\mathbf{s}_0$. We will then choose our coordinate system so that $\mathbf{s}_0 = \mathbf{0}$.

So,

$$\mathbf{s}(t) = -\frac{1}{2}gt^2\,\mathbf{j} + \mathbf{v}_0 t.$$

Projectile Motion

The vector equation of ideal projectile motion is

$$\mathbf{s}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t.$$

If the projectile's initial speed is v_0 and the angle θ is the projectile's **launch angle (firing angle, angle of elevation)**, then $\mathbf{v}_0 = v_0\cos\theta\,\mathbf{i} + v_0\sin\theta\,\mathbf{j}$. This gives

$$\mathbf{s}(t) = (v_0 t\cos\theta)\,\mathbf{i} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}.$$

Projectile Motion

If we write this vector equation out in parametric equations, we get

$$x(t) = v_0 t \cos\theta$$
$$y(t) = v_0 t \sin\theta - \frac{1}{2} g t^2.$$

Projectile Motion

If we want to find the flight time, we set y equal to zero and solve.

$$0 = v_0 t \sin\theta - \frac{1}{2}gt^2 = t\left(v_0 \sin\theta - \frac{1}{2}gt\right),$$

which gives us

$$t = 0 \quad \text{or} \quad t = \frac{2v_0 \sin\theta}{g}.$$

The flight time is the second of these two answers.

Projectile Motion

To find the range, we evaluate $x(t)$ at the time the projectile hits the ground.

$$\begin{aligned} x\left(\frac{2v_0 \sin\theta}{g}\right) &= (v_0 \cos\theta) \cdot \left(\frac{2v_0 \sin\theta}{g}\right) \\ &= \frac{v_0^2}{g} \cdot (2 \sin\theta \cos\theta) \\ &= \frac{v_0^2}{g} \sin 2\theta. \end{aligned}$$

Projectile Motion

The maximum height of the projectile is reached halfway through the flight time. To find the maximum height, we evaluate $y(t)$ at that time.

$$
\begin{aligned}
y\left(\frac{v_0 \sin\theta}{g}\right) &= (v_0 \sin\theta)\left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2 \\
&= \frac{(v_0 \sin\theta)^2}{g} - \frac{(v_0 \sin\theta)^2}{2g} \\
&= \frac{(v_0 \sin\theta)^2}{2g}.
\end{aligned}
$$

Height, Flight Time, and Range for Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle θ:

1. **Maximum height:** $y_{\text{max}} = \dfrac{(v_0 \sin\theta)^2}{2g}$
2. **Flight Time:** $t = \dfrac{2v_0 \sin\theta}{g}$
3. **Range:** $R = \dfrac{v_0^2}{g} \sin 2\theta$

Example

Example 4

Example

A projectile is fired with an initial speed of 500 m/s at an angle of elevation of 45 $^{\mathrm{o}}$.

1. When and how far away will the projectile strike?
2. How high overhead will the projectile be when it is 5 km downrange?
3. What is the greatest height reached by the projectile?

Example 4

Solution

1. The range is given by

$$R = \frac{v_0^2}{g} \sin 2\theta = \frac{500^2}{9.8} \sin(2 \cdot 45°) \approx 25{,}510.2 \text{ m}.$$

2. We need to find out the time when the projectile is 5 km downrange:

$$\begin{aligned} x(t) &= (v_0 \cos \theta) t \\ 5000 &= 500 \cos(45^\circ) t \\ t &= 10\sqrt{2}. \end{aligned}$$

Example 4

Solution (cont.)

2. Now we find the altitude at that time. Recall, $t = 10\sqrt{2}$ and $v_0 = 500$.

$$
\begin{aligned}
y(t) &= (v_0 \sin\theta)t - \frac{1}{2}gt^2 \\
&= (500\sin(45^\circ)) \cdot 10\sqrt{2} - \frac{1}{2} \cdot 9.8 \cdot (10\sqrt{2})^2 \\
&\approx 4020 \ m.
\end{aligned}
$$

Example 4

Solution (cont.)

3. Finally, the greatest height reached by the projectile is

$$
\begin{aligned}
y\text{max} &= \frac{(v_0 \sin\theta)^2}{2g} \\
&= \frac{(500 \sin 45^\circ)^2}{2 \cdot 9.8} \\
&\approx 6378 \ m.
\end{aligned}
$$

The Vector and Parametric Equations for Projectile Motion

If we fire our projectile from the point (x_0, y_0) instead of the origin, the position vector for the path of motion is

$$\mathbf{r}(t) = (x_0 + v_0 t \cos\theta)\,\mathbf{i} + \left(y_0 + v_0 t \sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}.$$

Projectile Motion with Wind Gusts

Projectile Motion with Wind Gusts

If there is another force acting on the projectile, we must adjust the initial velocity.

Example

Example 5

Example

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/s, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\,\mathbf{i}$ ft/s to the ball's velocity. (-8.8 ft/s=6 mph).

1. Find a vector equation (position vector) for the path of the baseball.
2. How high does the baseball go, and when does it reach its maximum height?
3. Assuming that the ball is not caught, find its range and flight time.

Example 5

Solution

We start with the acceleration due to gravity: $\mathbf{a} = -32\,\mathbf{j}$.
Integrating, we get

$$\mathbf{v} = -32t\,\mathbf{j} + \mathbf{v}_0.$$

The vector $\mathbf{v}_0$ comes in two parts: The part due to the bat hitting the ball and the part to due the gust of wind.

$$\begin{aligned}\mathbf{v}_0 &= (152\cos(20^\circ))\,\mathbf{i} + (152\sin(20^\circ)\,\mathbf{j}) - 8.8\,\mathbf{i}\\ &= (152\cos(20^\circ) - 8.8)\,\mathbf{i} + 152\sin(20^\circ)\,\mathbf{j}.\end{aligned}$$

So,

$$\mathbf{v} = (152\cos(20^\circ) - 8.8)\,\mathbf{i} + (152\sin(20^\circ) - 32t)\,\mathbf{j}.$$

Example 5

Solution (cont.)

Integrating again, we get the position vector

$$\mathbf{r} = (152\cos(20^\circ) - 8.8)\, t\, \mathbf{i} + \left(152\sin(20^\circ)t - 16t^2\right)\, \mathbf{j} + \mathbf{r}_0.$$

We are told the ball is 3 ft off the ground when it's hit, so $\mathbf{r}_0 = 3\,\mathbf{j}$. This gives the position vector of the ball as

$$\mathbf{r}(t) = (152\cos(20^\circ) - 8.8)\, t\, \mathbf{i} + \left(3 + (152\sin(20^\circ))t - 16t^2\right)\, \mathbf{j}.$$

Example 5

Solution (cont.)

The ball reaches its highest point when the **j**-component of its velocity is zero:

$$152\sin(20^\circ) - 32t = 0$$
$$t = \frac{152\sin(20^\circ)}{32}.$$

Example 5

Solution (cont.)

To find the maximum height of the projectile, we put this time into the **j**-component of its position vector:

$$y_{\max} = 3 + 152\sin(20^\circ)\left(\frac{152\sin(20^\circ)}{32}\right) - 16\left(\frac{152\sin(20^\circ)}{32}\right)^2$$

$$\approx 45.2 \text{ ft.}$$

Example 5

Solution (cont.)

The flight time is the time the **j**-component of its position vector is zero:

$$
\begin{aligned}
3 + (152 \sin(20^\circ))t - 16t^2 &= 0 \\
t &= \frac{19\sin(20^\circ)}{4} + \frac{\sqrt{3 + 361\sin^2(20^\circ)}}{4} \\
&\approx 3.3\ s.
\end{aligned}
$$

Example 5

Solution (cont.)

The range of the ball is the **i**-component of its position vector at this time:

$$R \approx (152\cos(20^\circ) - 8.8)\,(3.3) \approx 442 \ ft.$$