Arc Length and Curvature

William M. Faucette

University of West Georgia

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Outline

Arc Length for Vector Functions

Arc Length for Vector Functions

Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

Arc Length for Vector Functions

In a similar fashion, if we define a smooth curve using a vector-valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt.$$

Arc Length for Vector Functions

In three dimensions, if the vector-valued function is described by $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ over the same interval $[a, b]$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt.$$

Arc Length for Vector Functions

Theorem 3.4: Arc-Length Formulas

i. Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where $a \le t \le b$, the arc length of C over the interval

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt.$$

ii. Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where $a \le t \le b$, the arc length of C over the interval

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt.$$

Arc-Length Parameterization

Arc-Length Parameterization

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\, du.$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration.

Arc-Length Parameterization

Since $s(t)$ measures distance traveled as a function of time, $s'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$, we can differentiate both sides of the equation:

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\,du\right] \\ &= \frac{d}{dt}\left[\int_a^t \|\mathbf{r}'(u)\|\,du\right] \\ &= \|\mathbf{r}'(t)\|. \end{aligned}$$

Arc-Length Parameterization

Theorem 3.5: Arc-Length Function

Let $\mathbf{r}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| \, du.$$

Further, $\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0$. If $\|\mathbf{r}'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

Arc-Length Parameterization

A useful application of this theorem is to find an alternative parameterization of a given curve, called an arc-length parameterization. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle, \quad 0 \le t \le 2\pi.$$

that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization

$$\mathbf{r}(t) = \langle 3\cos 4t, 3\sin 4t \rangle, \quad 0 \le t \le \pi/2.$$

The new parameterization still defines a circle of radius 3, but now we need only use the values $0 \le t \le \pi/2$ to traverse the circle once.

Arc-Length Parameterization

One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example

Example 1

Example

Find the arc-length function for the helix

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle, \quad t \geq 0.$$

Example 1

Solution

We are given

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad t \geq 0.$$

We first compute $\mathbf{r}'(t)$ and its length:

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -3\sin t, 3\cos t, 4\rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} \\
&= \sqrt{9\sin^2 t + 9\cos^2 t + 16} \\
&= \sqrt{9(\sin^2 t + \cos^2 t) + 16} \\
&= \sqrt{9 + 16} = \sqrt{25} = 5.
\end{aligned}$$

Example 1

Solution (cont.)

So, we have

$$
\begin{aligned}
s(t) &= \int_0^t \|\mathbf{r}'(u)\|\, du \\
&= \int_0^t 5\, du \\
&= 5t.
\end{aligned}
$$

So, we have $s = 5t$.

Example 1

Solution (cont.)

Using the change of parametrization $s = 5t$ in the original parametrization, we get

$$\mathbf{r}(s) = \left\langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5} \right\rangle, \quad s \geq 0.$$

Curvature

Curvature

Definition

Let C be a smooth curve in the plane or in space given by $\mathbf{r}(s)$, where s is the arc-length parameter. The **curvature** κ at s is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \mathbf{T}'(s) \right\|.$$

Curvature

We would like to be able to compute the curvature without computing the arc length parametrization. For that, we have the following theorem.

Theorem 3.6: Alternative Formulas for Curvature

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$

Curvature

Proof

From the chain rule, we have

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

. Dividing by ds/dt, we get

$$\mathbf{T}'(s) = \frac{\mathbf{T}'(t)}{ds/dt}$$

.

Curvature

Proof.

Since $ds/dt = \|\mathbf{r}'(t)\|$, this gives

$$\kappa = \left\|\mathbf{T}'(s)\right\| = \frac{\|\mathbf{T}(t)\|}{\|\mathbf{r}'(t)\|}.$$

□

Curvature

We have another useful formula for curvature for curves in space.

Theorem 3.6: Alternative Formulas for Curvature

If C is a three-dimensional curve, then the curvature can be given by the for

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Curvature

Proof.

In the case of a three-dimensional curve, we start with the formulas $\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$ and $ds/dt = \|\mathbf{r}'(t)\|$. Therefore, $\mathbf{r}'(t) = (ds/dt)\mathbf{T}(t)$.

We can take the derivative of this function using the scalar product formula:

$$\mathbf{r}''(t) = \frac{d^2 s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

Curvature

Proof (cont.)

Recall that $\mathbf{r}'(t) = (ds/dt)\mathbf{T}(t)$ and

$$\mathbf{r}''(t) = \frac{d^2 s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t).$$

Curvature

Proof (cont.)

Substituting, we get

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \frac{ds}{dt}\mathbf{T}(t) \times \left(\frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t) \right) \\
&= \frac{ds}{dt}\frac{d^2s}{dt^2}\mathbf{T}(t) \times \mathbf{T}(t) + \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \mathbf{T}'(t) \\
&= \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \mathbf{T}'(t).
\end{aligned}$$

since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$.

Curvature

Proof (cont.)

Using the chain rule one more time, we have $d\mathbf{T}/dt = (d\mathbf{T}/ds)(ds/dt)$. Substituting this into the last equation gives us

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \mathbf{T}'(t) \\
&= \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\
&= \left(\frac{ds}{dt}\right)^3 \mathbf{T}(t) \times \frac{d\mathbf{T}}{ds}.
\end{aligned}$$

Curvature

Proof (cont.)

Since $\mathbf{T}$ is a unit vector, we have $\mathbf{T} \cdot \mathbf{T} = 1$. Taking the derivative with respect to s using the product rule, and dividing by 2, we get $\mathbf{T} \cdot d\mathbf{T}/ds = 0$. So, $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$.

Hence

$$\left\| \mathbf{T}(t) \times \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}(t)\| \left\| \frac{d\mathbf{T}}{ds} \right\| \sin \frac{\pi}{2} = \left\| \frac{d\mathbf{T}}{ds} \right\| = \kappa$$

Curvature

Proof (cont.)

Putting this altogether, we have

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left|\frac{ds}{dt}\right|^3 \left\|\mathbf{T}(t) \times \frac{d\mathbf{T}}{ds}\right\| = \left|\frac{ds}{dt}\right|^3 \kappa.$$

Solving for κ and using the fact that $\|\mathbf{r}'(t)\| = ds/dt$, we get

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

This gives the desired result. □

Curvature

We have another useful formula for curvature for curves in the plane parametrized by x.

Theorem 3.6: Alternative Formulas for Curvature

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

Curvature

Proof

We treat this curve as a curve in space parametrized by x with z-component identically zero. This gives us the curve as

$$\mathbf{r}(t) = x\,\mathbf{i} + f(x)\,\mathbf{j}.$$

Curvature

Proof (cont.)

Now we compute:

$$\begin{aligned}
\mathbf{r}'(t) &= \mathbf{i} + f'(x)\,\mathbf{j} \\
\mathbf{r}''(t) &= f''(x)\,\mathbf{j} \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= (\mathbf{i} + f'(x)\,\mathbf{j}) \times f''(x)\,\mathbf{j} \\
&= f''(x)\,\mathbf{i} \times \mathbf{j} + f'(x)f''(x)\,\mathbf{j} \times \mathbf{j} \\
&= f''(x)\,\mathbf{k}.
\end{aligned}$$

since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$.

Curvature

Proof (cont.)

Using the previous formula, we get

$$\begin{aligned}\kappa &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \\ &= \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}^3} \\ &= \frac{|y''|}{(1 + (y')^2)^{3/2}},\end{aligned}$$

which is the formula we want. □

Normal and Binormal Vectors

Normal and Binormal Vectors

Definition

Let C be a three-dimensional smooth curve represented by $\mathbf{r}$ over an open interval I. If $\mathbf{T}'(t) \neq 0$, then the **principal unit normal vector at** t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

The binormal vector at t is defined as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t),$$

where $\mathbf{T}(t)$ is the unit tangent vector.

Normal and Binormal Vectors

We remark that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are mutually orthogonal unit vectors that form a right hand coordinate system at each point on the curve.

We also remark that

$$\frac{d\mathbf{T}}{ds} = \left\| \frac{d\mathbf{T}}{ds} \right\| \cdot \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} = \kappa \, \mathbf{N},$$

Example

Example

The position function for a particle is

$$\mathbf{r}(t) = a\cos(\omega t)\,\mathbf{i} + a\sin(\omega t)\,\mathbf{j}, \quad a, \omega > 0.$$

Find the unit tangent vector, the principal unit normal vector, and the binormal vector at each point of the curve traced out by the particle.

Solution

We start our nasty computations:

$$\begin{aligned}
\mathbf{r}(t) &= a\cos(\omega t)\,\mathbf{i} + a\sin(\omega t)\,\mathbf{j} \\
\mathbf{r}'(t) &= -a\omega\sin(\omega t)\,\mathbf{i} + a\omega\cos(\omega t)\,\mathbf{j} \\
\|\mathbf{r}'(t)\| &= \sqrt{(-a\omega\sin(\omega t))^2 + (a\omega\cos(\omega t))^2} \\
&= \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2\cos^2(\omega t)} \\
&= \sqrt{a^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} \\
&= \sqrt{a^2\omega^2} = a\omega.
\end{aligned}$$

Normal and Binormal Vectors

Solution (cont.)

So, we have

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{-a\omega\sin(\omega t)\,\mathbf{i} + a\omega\cos(\omega t)\,\mathbf{j}}{a\omega} \\ &= -\sin(\omega t)\,\mathbf{i} + \cos(\omega t)\,\mathbf{j}.\end{aligned}$$

Normal and Binormal Vectors

Solution (cont.)

Continuing our computations,

$$\begin{aligned}
\mathbf{T}(t) &= -\sin(\omega t)\,\mathbf{i} + \cos(\omega t)\,\mathbf{j} \\
\mathbf{T}'(t) &= -\omega\cos(\omega t)\,\mathbf{i} - \omega\sin(\omega t)\,\mathbf{j} \\
\|\mathbf{T}'(t)\| &= \sqrt{(-\omega\cos(\omega t))^2 + (-\omega\sin(\omega t))^2} \\
&= \sqrt{(\omega^2\cos^2(\omega t) + \omega^2\sin^2(\omega t)} \\
&= \sqrt{\omega^2(\cos^2(\omega t) + \sin^2(\omega t))} \\
&= \sqrt{\omega^2} = \omega.
\end{aligned}$$

Normal and Binormal Vectors

Solution (cont.)

So, we have

$$\begin{aligned}\mathbf{N}(t) &= \frac{T'(t)}{\|T'(t)\|} \\ &= \frac{-\omega\cos(\omega t)\,\mathbf{i} - \omega\sin(\omega t)\,\mathbf{j}}{\omega} \\ &= -\cos(\omega t)\,\mathbf{i} - \sin(\omega t)\,\mathbf{j}.\end{aligned}$$

Normal and Binormal Vectors

Solution (cont.)

Finally, we have

$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= (-\sin(\omega t)\,\mathbf{i} + \cos(\omega t)\,\mathbf{j}) \times (-\cos(\omega t)\,\mathbf{i} - \sin(\omega t)\,\mathbf{j}) \\
&= \sin^2(\omega t)\,\mathbf{i} \times \mathbf{j} - \cos^2(\omega t)\,\mathbf{j} \times \mathbf{i} \\
&= [\sin^2(\omega t) + \cos^2(\omega t)]\,\mathbf{k} \\
&= \mathbf{k}.
\end{aligned}$$

The Osculating Circle

The Osculating Circle

The plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$ forms the **osculating plane** of C at any point P on the curve.

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $1/r$. We call r the **radius of curvature** of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the **osculating circle** of C at P.

Example

The Osculating Circle

Example

Find the equation of the osculating circle of the helix defined by the function $y = x^3 - 3x + 1$ at $x = 1$.

The Osculating Circle

Solution

First, let's calculate the curvature at $x = 1$:
We compute

$$\begin{aligned}\kappa = &= \frac{|y''|}{(1+(y')^2)^{3/2}} \\ &= \frac{|6x|}{(1+(3x^2-3)^2)^{3/2}}\end{aligned}$$

At $x = 1$, this gives $\kappa = 6$.

Therefore, the radius of the osculating circle is given by
$R = 1/\kappa = 1/6$.

The Osculating Circle

Solution (cont.)

Next, we then calculate the coordinates of the center of the circle.

When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -5/6)$

The Osculating Circle

Solution (cont.)

The formula for a circle with radius r and center (h, k) is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

therefore the equation of the osculating circle is

$$(x-1)^2 + \left(y + \frac{5}{6}\right)^2 = \frac{1}{36}.$$

A sketch of the curve and the osculating circle are on the next slide.

The Osculating Circle

Solution (cont.)

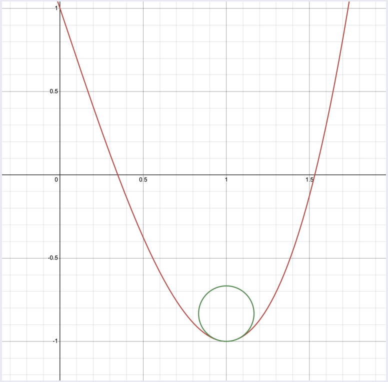

Figure: Osculating Circle