Vector-Valued Functions and Space Curves

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Fall 2025

Outline

Curves in Space and Their Tangents

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A **curve in space** is given by parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

for t in some interval I.

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A curve in space can be represented in vector form:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

is the position vector of a point on the curve at time t.

In this form, the functions $x(t)$, $y(t)$, $z(t)$ are **component functions**.

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This is an example of a **vector-valued function**. A vector valued function assigns a vector to each point in a domain set D.

The function we have studied up to now are called **scalar functions.** These functions assign a number to each point in a domain set D.

Example

Example 1

Example

The function

$$\mathbf{r}(t) = (t+1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j}$$

describes the curve $y = x^2 - 2x$.

Graphing Vector-Valued Functions

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The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ consists of the set of all $(t, \mathbf{r}(t))$, and the path it traces is called a **plane curve**.

The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ consists of the set of all $(t, \mathbf{r}(t))$, and the path it traces is called a **space curve**.

Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Limits

Limits

We now take a look at the **limit of a vector-valued function**. This is important to understand to study the calculus of vector-valued functions.

Definition

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches t_0, written

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

provided

$$\lim_{t \to t_0} \|\mathbf{r}(t) - \mathbf{L}\| = 0.$$

Limits

We omit the proof, but the limit of a vector valued function $\mathbf{r}$ is gotten by taking as component functions the limits of the respective component functions of $\mathbf{r}$.

If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then

$$\lim_{t \to t_0} \mathbf{r}(t) = \left(\lim_{t \to t_0} x(t) \right) \mathbf{i} + \left(\lim_{t \to t_0} y(t) \right) \mathbf{j} + \left(\lim_{t \to t_0} z(t) \right) \mathbf{k}$$

Example

Example 2

Example

Compute

$$\lim_{t\to -1}\left[t^3\,\mathbf{i}+\left(\sin\frac{\pi}{2}t\right)\,\mathbf{j}+(\ln(t+2))\,\mathbf{k}\right]$$

Example 2

Solution

$$\begin{aligned}
\lim_{t\to -1}\left[t^3\,\mathbf{i}+\left(\sin\frac{\pi}{2}t\right)\,\mathbf{j}+(\ln(t+2))\,\mathbf{k}\right]
&=\left[\lim_{t\to -1}t^3\,\mathbf{i}+\lim_{t\to -1}\left(\sin\frac{\pi}{2}t\right)\,\mathbf{j}+\lim_{t\to -1}(\ln(t+2))\,\mathbf{k}\right]\\
&=\left[(-1)^3\,\mathbf{i}+\left(\sin\frac{\pi}{2}(-1)\right)\,\mathbf{j}+\ln((-1)+2)\,\mathbf{k}\right]\\
&=-\mathbf{i}-\mathbf{j}.
\end{aligned}$$

Continuity

Continuity

The definition of continuity for vector valued functions also mirrors the definition of continuity you saw in Calculus 1.

Definition

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Example

Example 3

Example

The function

$$\mathbf{r}(t) = (t+1)\,\mathbf{i} + (t^2+1)\,\mathbf{j} + (t-1)\,\mathbf{k}$$

is continuous at all $t \in \mathbb{R}$ because each component function is continuous everywhere.