Vectors in Three Dimensions

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Fall 2025

Outline

Three-Dimensional Coordinate Systems

Three-Dimensional Coordinate Systems

Just as you can put Cartesian (or rectangular) coordinates on the plane using an x-axis and a y-axis which are perpendicular, you can also put **Cartesian** (or **rectangular**) **coordinates** on 3-dimensional space. Here, you use an x-axis, a y-axis, and a z-axis.

The y-axis goes left and right. The z-axis goes up and down. The x-axis goes forward and backward out of the plane of the drawing. Because of this, the x-axis is drawn at an angle to provide perspective.

The axes are labeled so that the coordinate system is a **right hand coordinate system**.

See the sketch on the next slide.

Three-Dimensional Coordinate Systems

Figure: Three-Dimensional Coordinates

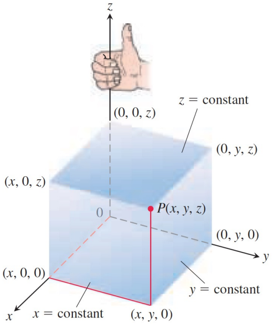


FIGURE 12.1 The Cartesian coordinate system is right-handed.

Three-Dimensional Coordinate Systems

The coordinate system consists of three **coordinate planes**.

The set of points $x = 0$ is the yz-**coordinate plane**. It is the plane of the paper.

The set of points $y = 0$ is the xz-**coordinate plane**. It is a vertical plane coming out toward you from the plane of the paper.

The set of points $z = 0$ is the xy-**coordinate plane**. It is a horizontal plane coming out toward you from the plane of the paper.

Three-Dimensional Coordinate Systems

The coordinate planes divide space into eight **octants**. Only one octant is named. The **first octant** is the set of all points in space where $x > 0$, $y > 0$, and $z > 0$. It is the octant you are looking at when you draw the axes as in the preceding picture.

Three-Dimensional Coordinate Systems

A point in space is given by an ordered triple of numbers (a, b, c).

You go a units in the x direction. If $a > 0$, that is coming out toward you. If $a < 0$, that is going backward away from you.

You go b units in the y direction. If $b > 0$, that is going to the right. If $b < 0$, that is going to the left.

You go c units in the z direction. If $c > 0$, that is going up. If $c < 0$, is going down.

Linear Equations in Three Variables

Linear Equations in Three Variables

A plane in space is given by one nonzero linear equation, $ax + by + cz = d$, where at least one of a, b, and c is not zero.

A line in space is given by two nonzero linear equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

This is simply the line where the two planes given by these two equations intersect.

Linear Equations in Three Variables

It is possible that two linear equations define the same plane. In this case, the planes are **coincident**. This happens when (a_1, b_1, c_1, d_1) is a constant multiple of (a_2, b_2, c_2, d_2).

It is possible that two linear equations define planes that do not meet. In this case, the planes are **parallel**. This happens when (a_1, b_1, c_1) is a constant multiple of (a_2, b_2, c_2), but (a_1, b_1, c_1, d_1) is not a constant multiple of (a_2, b_2, c_2, d_2).

Regions in Space

Regions in Space

A region given by $ax + by + cz \geq d$ or $ax + by + cz \leq d$ (once again with at least one of a, b, and c is not zero) is a **closed half-space**.

A region given by $ax + by + cz > d$ or $ax + by + cz < d$ (once again with at least one of a, b, and c is not zero) is an **open half-space**.

Other regions can also be defined by equalities and inequalities.

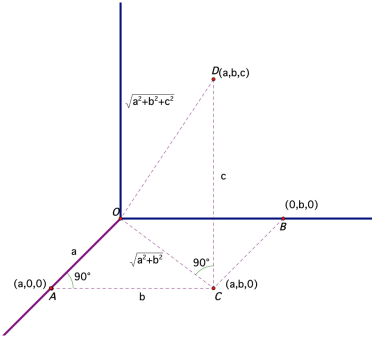
Distance and Spheres in Space

Distance and Spheres in Space

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This formula is gotten by applying the Pythagorean Theorem twice to two perpendicular right triangles.



Distance and Spheres in Space

The standard form of the equation of a sphere with center (a, b, c) and radius $r > 0$ is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

This says the distance from the point (x, y, z) to the point (a, b, c) is r.

Example

Example

Example

Find the center C and radius r for the sphere

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9$$

Example

Solution

What you have to do here is to complete the square in each of x, y, and z.

$$\begin{aligned}
&2x^2 + 2y^2 + 2z^2 + x + y + z = 9 \\
&x^2 + y^2 + z^2 + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = \frac{9}{2} \\
&\left(x^2 + \frac{1}{2}x + \quad\right) + \left(y^2 + \frac{1}{2}y + \quad\right) + \left(z^2 + \frac{1}{2}z + \quad\right) = \frac{9}{2}
\end{aligned}$$

Example

Solution (cont.)

$$\left(x^2+\frac{1}{2}x+\quad\right)+\left(y^2+\frac{1}{2}y+\quad\right)+\left(z^2+\frac{1}{2}z+\quad\right)=\frac{9}{2}$$

$$\left(x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2\right)+\left(y^2+\frac{1}{2}y+\left(\frac{1}{4}\right)^2\right)$$

$$+\left(z^2+\frac{1}{2}z+\left(\frac{1}{4}\right)^2\right)=\frac{9}{2}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{75}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\left(\frac{5\sqrt{3}}{4}\right)^2.$$

Example

Solution (cont.)

From the equation

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\left(\frac{5\sqrt{3}}{4}\right)^2,$$

we see that this is an equation of the sphere with radius $\frac{5\sqrt{3}}{4}$ and center $\left(-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$.

Working with Vectors in $\mathbb{R}^3$

Vector Representations

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow. Three-dimensional vectors can also be represented in component form.

Vector Representations

The notation $\mathbf{v} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\mathbf{0} = \langle 0, 0, 0 \rangle$. So, for example, the three dimensional vector $\mathbf{v} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector Representations

Just as in the plane, vectors can be given by the initial point and terminal point.

Let **v** be a vector with initial point (x_i, y_i, z_i) and terminal point (x_t, y_t, z_t). Then we can express **v** in component form as

$$\mathbf{v} = \langle x_t - x_i, y_t - y_i, z_t - z_i \rangle.$$

Vector Operations

Vector Operations

Definition

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$, and let k be a scalar.
Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$
Vector addition:

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle \\ &= \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \end{aligned}$$

Vector subtraction:

$$\begin{aligned} \mathbf{v} - \mathbf{w} &= \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle \\ &= \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \end{aligned}$$

Vector Operations

Definition

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$, and let k be a scalar.

Vector magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

Unit vector in the direction of v:

$$\frac{1}{\|\mathbf{v}\|}\,\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\,\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \right\rangle$$

if $\mathbf{v} \neq \mathbf{0}$.