Vectors In The Plane

William M. Faucette

University of West Georgia

Fall 2025

Outline

Vector Representation

Vector Representation

Some quantities, such as or force, are defined in terms of both size (also called *magnitude*) and direction. A quantity that has magnitude and direction is called a **vector**. In this text, we denote vectors by boldface letters, such as **v**.

Definition

A **vector** is a quantity that has both magnitude and direction. Vectors are said to be **equal** if they have the same magnitude and direction.

Example 2.1

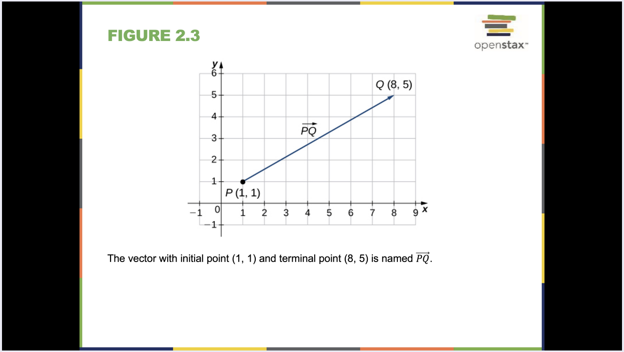
Example

Sketch a vector in the plane from initial point $P(1, 1)$ to terminal point $Q(8, 5)$.

Vector Representation

Solution

See Figure 2.3. Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.



Combining Vectors

Combining Vectors

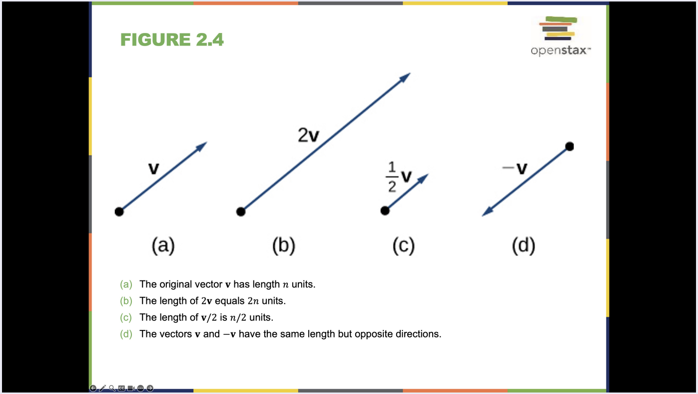
Definition

The product $k\mathbf{v}$ of a vector $\mathbf{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\mathbf{v}$, and with a direction that is the same as the direction of $\mathbf{v}$ if $k > 0$, and opposite the direction of $\mathbf{v}$ if $k < 0$. This is called **scalar multiplication**. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then $k\mathbf{v} = \mathbf{0}$.

Vector Representation

Illustration

See Figure 2.4.



Combining Vectors

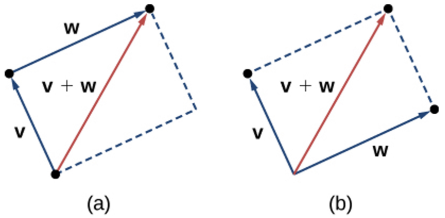
Definition

The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ can be constructed graphically by placing the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$. Then, the vector sum, $\mathbf{v} + \mathbf{w}$, is the vector with an initial point that coincides with the initial point of $\mathbf{v}$ and has a terminal point that coincides with the terminal point of $\mathbf{w}$. This operation is known as **vector addition**. (See Figure 2.5 on the next slide.)

Vector Representation

Illustration

FIGURE 2.5



(a) When adding vectors by the triangle method, the initial point of $\mathbf{w}$ is the terminal point of $\mathbf{v}$.

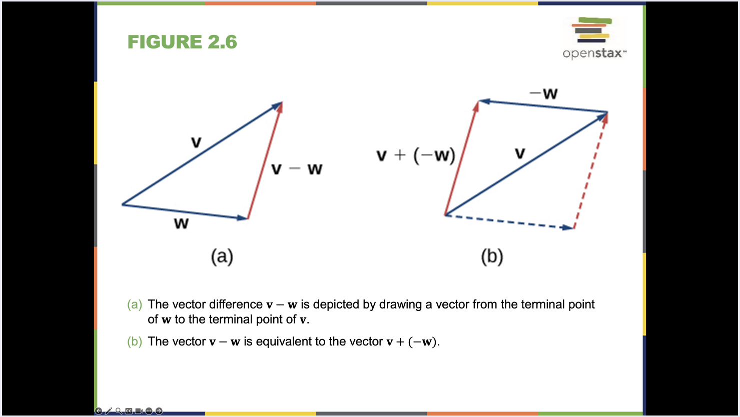
(b) When adding vectors by the parallelogram method, the vectors $\mathbf{v}$ and $\mathbf{w}$ have the same initial point.

Combining Vectors

Definition

It is also appropriate here to discuss vector subtraction. We define $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$. The vector $\mathbf{v} - \mathbf{w}$ is called the **vector difference**. Graphically, the vector $\mathbf{v} - \mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$. (See Figure 2.6 on the next slide.)

Illustration



Combining Vectors

The initial point of $\mathbf{v} + \mathbf{w}$ is the initial point of $\mathbf{v}$. The terminal point of $\mathbf{v} + \mathbf{w}$ is the terminal point of $\mathbf{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have the **triangle inequality**.

The Triangle Inequality

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

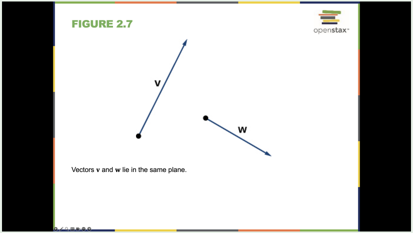
There is one case, however, when the resultant vector $\mathbf{v} + \mathbf{w}$ has the same magnitude as the sum of the magnitudes of $\mathbf{v}$ and $\mathbf{w}$. This happens only when $\mathbf{v}$ and $\mathbf{w}$ have the same direction.

Example 2.2

Example

Given the vectors **v** and **w** shown in Figure 2.7 on the next slide, sketch the vectors

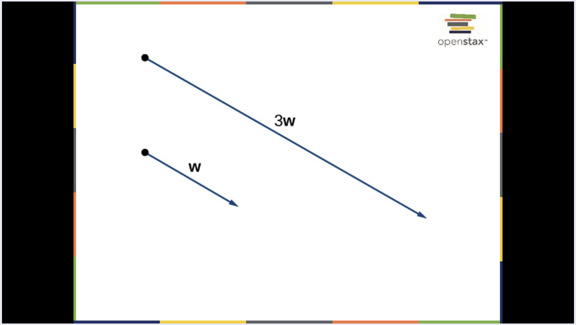
a. $3\mathbf{w}$
b. $\mathbf{v} + \mathbf{w}$
c. $2\mathbf{v} - \mathbf{w}$



Vector Representation

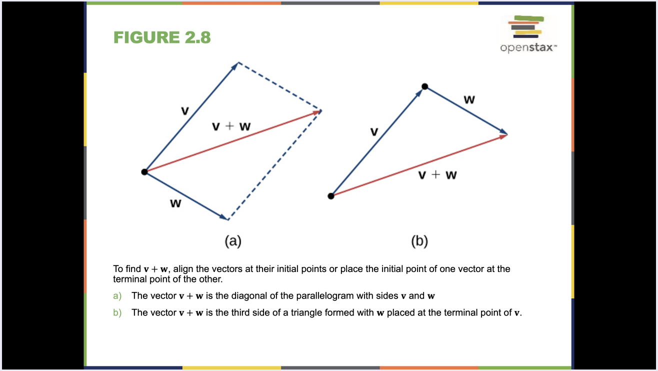
Solution

- a The vector 3**w** has the same direction as **w**; it is three times as long as **w**.



Solution (cont.)

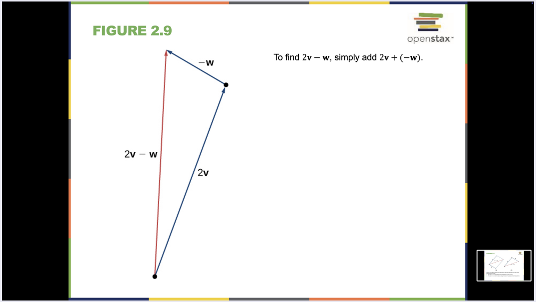
b. Use either addition method to find $\mathbf{v} + \mathbf{w}$.



Vector Representation

Solution (cont.)

c. To find $2\mathbf{v} - \mathbf{w}$, we can first rewrite the expression as $2\mathbf{v} + (-\mathbf{w})$. Then we can draw the vector $-\mathbf{w}$, then add it to the vector $2\mathbf{v}$.



Vector Components

Vector Components

Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 2.3

Example

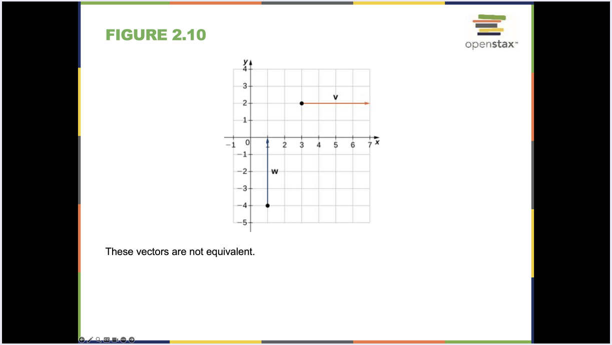
Are $\mathbf{v}$ and $\mathbf{w}$ equal vectors?

a. $\mathbf{v}$ has initial point $(3, 2)$ and terminal point $(7, 2)$
   $\mathbf{w}$ has initial point $(1, -4)$ and terminal point $(1, 0)$
b. $\mathbf{v}$ has initial point $(0, 0)$ and terminal point $(1, 1)$
   $\mathbf{w}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

Example 2.3

Solution

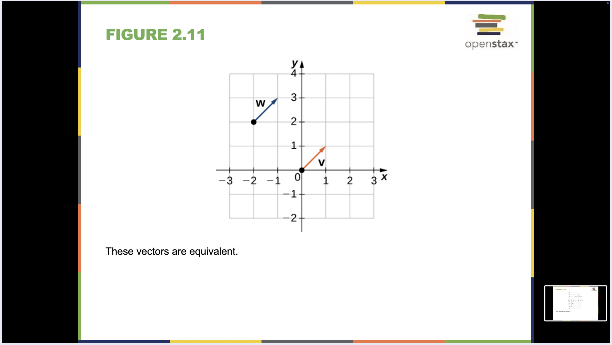
a The vectors are each 4 units long, but they are oriented in different directions. So **v** and **w** are not equivalent (Figure 2.10).



Example 2.3

Solution (cont.)

b Based on Figure 2.11, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so **v** and **w** are equal.



Vector Components

We call a vector with its initial point at the origin a **standard-position vector**. Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector $\mathbf{v}$ has its initial point at the origin and its terminal point at (x, y), we write the vector in component form as

$$\mathbf{v} = \langle x, y \rangle.$$

When a vector is written in component form like this, the scalars x and y are called the **components** of $\mathbf{v}$.

Vector Components

Definition

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$\mathbf{v} = \langle x, y \rangle.$$

The scalars x and y are called the **components** of $\mathbf{v}$.

Vector Components

Rule

Let $\mathbf{v}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express $\mathbf{v}$ in component form as

$$\mathbf{v} = \langle x_t - x_i, y_t - y_i \rangle.$$

Example 2.4

Example

Express vector $\mathbf{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Example 2.4

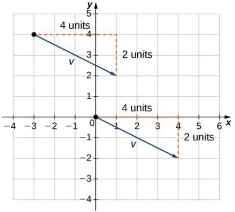
Solution

a. Geometric
   1. Sketch the vector in the coordinate plane (Figure 2.12).
   2. The terminal point is 4 units to the right and 2 units down from the initial point.
   3. Find the point that is 4 units to the right and 2 units down from the origin.
   4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:

Example 2.4

Solution (cont.)

FIGURE 2.12



These vectors are equivalent.

Example 2.4

Solution (cont.)

b Algebraic

In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

Example 2.4

Solution (cont.)

b Algebraic (cont.)
So, in component form,

$$\begin{aligned}\mathbf{v} &= \langle x_t - x_i, y_t - y_i \rangle \\ &= \langle 1 - (-3), 2 - 4 \rangle \\ &= \langle 4, -2 \rangle.\end{aligned}$$

Vector Components

The magnitude of vector $\mathbf{v} = \langle x, y \rangle$ is denoted $\|\mathbf{v}\|$, or $|\mathbf{v}|$, and can be computed using the formula

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}$$

Based on this formula, it is clear that for any vector $\mathbf{v}$, $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$, the zero vector.

Vector Operations

Vector Operations

Definition

Let $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$, and let k be a scalar.
Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1 \rangle$
Vector addition:

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle \\ &= \langle x_1 + x_2, y_1 + y_2 \rangle \end{aligned}$$

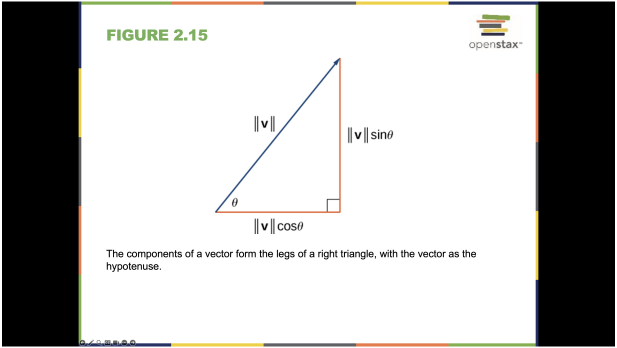
Vector Operations

Theorem 2.1: Properties of Vector Operations

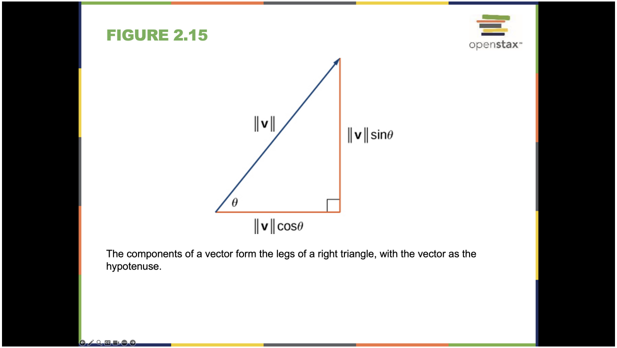
i.	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
ii.	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property
iii.	$\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
iv.	$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
v.	$r(s\mathbf{u}) = (rs)\mathbf{u}$	Associativity of scalar multiplication
vi.	$(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$	Distributive property
vii.	$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$	Distributive property
viii.	$1\mathbf{u} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$	Identity and zero properties

Vector Operations

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (See Figure 2.15).



Vector Operations



Consider the angle θ formed by the vector **v** and the positive x-axis. We can see from the triangle that the components of vector **v** are $\langle \|\mathbf{v}\| \cos\theta, \|\mathbf{v}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

Unit Vectors

Unit Vectors

A **unit vector** is a vector with magnitude 1.

For any nonzero vector $\mathbf{v}$, we can use scalar multiplication to find a unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

Unit Vectors

The **standard unit vectors** are the vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

By applying the properties of vectors, it is possible to express any vector in terms of $\mathbf{i}$ and $\mathbf{j}$ in what we call a linear combination:

$$\mathbf{v} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x\langle 1, 0 \rangle + y\langle 0, 1 \rangle = x\,\mathbf{i} + y\,\mathbf{j}.$$

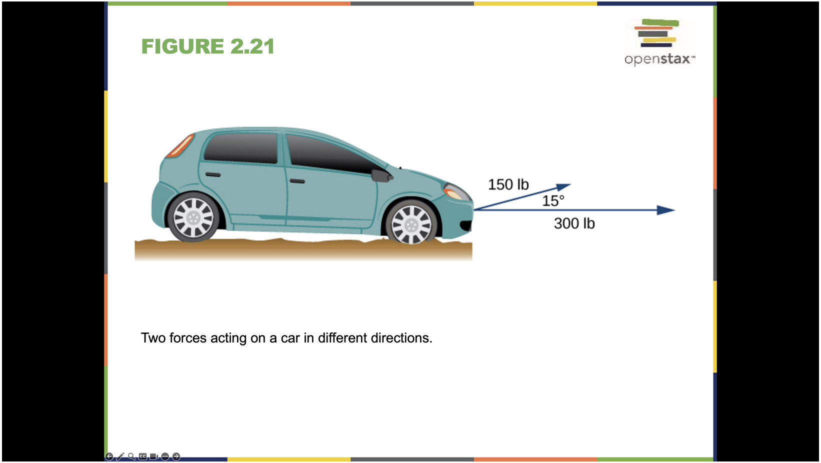
Finding Resultant Force

Finding Resultant Force

Jane's car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck's trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of 300 lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of 150 lb on the car. These forces can be represented by vectors. The angle between these vectors is 15°. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive x-axis.

See the sketch in Figure 2.21 on the next slide.

Finding Resultant Force



Finding Resultant Force

Solution

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive x-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$.

It can be expressed as $\langle 300, 0 \rangle$ or $300\,\mathbf{i}$.

Finding Resultant Force

Solution

The second vector has magnitude 150 and makes an angle of 15°with the first, so we can express it as

$$\langle 150\cos(15^\circ), 150\sin(15^\circ)\rangle$$

or

$$150\cos(15^\circ)\,\mathbf{i} + 150\sin(15^\circ)\,\mathbf{j}.$$

Finding Resultant Force

Solution (cont.)

Then, the sum of the vectors, or resultant vector, is

$$\begin{aligned} \mathbf{r} &= \langle 300, 0 \rangle + \langle 150\cos(15^\circ), 150\sin(15^\circ) \rangle \\ &= \langle 300 + 150\cos(15^\circ), 150\sin(15^\circ) \rangle. \end{aligned}$$

and we have

$$\begin{aligned} \|\mathbf{r}\| &= \sqrt{[300 + 150\cos(15^\circ)]^2 + [150\sin(15^\circ)]^2} \\ &\approx 446.6. \end{aligned}$$

Finding Resultant Force

Solution (cont.)

The angle θ made by $\mathbf{r}$ and the positive x-axis has

$$\tan\theta = \frac{150\sin 15^\circ}{300 + 150\cos 15^\circ} \approx 0.09,$$

so $\theta \approx \tan^{-1}(0.009) \approx 5^\circ$, which means the resultant force $\mathbf{r}$ has an angle of 5° above the horizontal axis.