Study Guide for MATH 2654 Final Examination

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Chapter 2

Know about **three-dimensional coordinates systems**, including rectangular (or Cartesian), cylindrical, and spherical coordinates.

The **distance between** $P_1(x_1, y_1, z_1)$ **and** $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

An **equation of a sphere with center** $C(h, k, \ell)$ **and radius** r is

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2.$$

Know what a **vector** is, including both angle bracket notation and $\vec{i}$, $\vec{j}$ (and $\vec{k}$ in space) notation. Know what a **displacement vector** is.

If the **initial point** of a vector $\mathbf{v}$ is $A = (a_1, a_2)$ and the **terminal point** of a vector $\mathbf{v}$ is $B = (b_1, b_2)$, the displacement vector, $\overrightarrow{AB}$ is

$$< b_1 - a_1, b_2 - a_2, b_3 - a_3 >= (b_1 - a_1)\,\vec{i} + (b_2 - a_2)\,\vec{j} + (b_3 - a_3)\,\vec{k}.$$

If $\mathbf{v} =< x_1, y_1, z_1 >$ and $\mathbf{w} =< x_2, y_2, z_2 >$, then the **sum** of the two vectors is

$$\mathbf{v} + \mathbf{w} =< x_1 + x_2, y_1 + y_2, z_1 + z_2 > .$$

If λ is a scalar, the **product** of λ and $\mathbf{v}$ is

$$\lambda\mathbf{v} =< \lambda x_1, \lambda y_1, \lambda z_1 > .$$

This is called **scalar multiplication**. The vector $-\mathbf{v} = (-1)\mathbf{v}$.

We define the **difference** of two vectors by

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w} =< x_1 - x_2, y_1 - y_2, z_1 - z_2 > .$$

You should also know how to add and scalar multiply vectors geometrically. This means you need to know the **Triangle Law** and the **Parallelogram Law**. The sum of two vectors is also called the **resultant**.

If $\mathbf{v} =< x_1, y_1, z_1 >$, then the x-**component** of $\mathbf{v}$ is the scalar x_1, the y-**component** of $\mathbf{v}$ is the scalar y_1, and the z-**component**of $\mathbf{v}$ is the scalar z_1.

There is a one-to-one correspondence between points and vectors. If the point (x, y) is in the plane, the corresponding **position vector** is $< x, y >$. If the point (x, y, z) is in space, the corresponding **position vector** is $< x, y, z >$. This is just the vector having initial point $(0, 0)$ (or $(0, 0, 0)$ in space) and terminal point (x, y) (or (x, y, z) in space).

The **length** or **magnitude** of the two-dimensional vector $\mathbf{v} =< x, y >$ is

$$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$

Similarly, the **length** or **magnitude** of the three-dimensional vector $\mathbf{v} =< x, y, z >$ is

$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}.$$

Length is also denoted $\|\mathbf{v}\|$. The length of a vector $\mathbf{v}$ is zero if and only if the vector $\mathbf{v}$ equals the zero vector, $\mathbf{0} = (0, 0)$ (or $\mathbf{0} = (0, 0, 0)$ in space).

Properties of Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are vectors and c and d are scalars then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} & \qquad (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \mathbf{a} + (\mathbf{b} + \mathbf{c}) \\ \mathbf{a} + \mathbf{0} &= \mathbf{a} & \qquad \mathbf{a} + (-\mathbf{a}) &= \mathbf{0} \\ c(\mathbf{a} + \mathbf{b}) &= c\mathbf{a} + c\mathbf{b} & \qquad (c + d)\mathbf{a} &= c\mathbf{a} + d\mathbf{a} \\ (cd)\mathbf{a} &= c(d\mathbf{a}) & \qquad 1\mathbf{a} &= \mathbf{a}. \end{aligned}$$

There are special vectors in the plane, $\vec{i} =< 1, 0 >$ and $\vec{j} =< 0, 1 >$, called the **standard basis vectors in** $\mathbb{R}^2$**.** Each vector in the plane can be written uniquely as a linear combination of $\vec{i}$ and $\vec{j}$: $< x, y >= x\,\vec{i} + y\,\vec{j}$.

Analogously, there are special vectors in space, $\vec{i} =< 1, 0, 0 >$, $\vec{j} =< 0, 1, 0 >$ and $\vec{k} =< 0, 0, 1 >$, called the **standard basis vectors in** $\mathbb{R}^3$**.** Each vector in space can be written uniquely as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$: $< x, y, z >= x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.

If $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$, the **dot product** (or **scalar product** or **inner product**) is defined by

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Properties of the Dot Product

$$\begin{aligned} &\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2 & \qquad &\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \\ &\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} & \qquad &(c\mathbf{a}) \bullet \mathbf{b} = c(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet (c\mathbf{b}) \\ &\mathbf{0} \bullet \mathbf{a} = 0 & & \end{aligned}$$

The most important property of the dot product is this: If **a** and **b** are vectors, then

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta.$$

where θ is the angle between **a** and **b**.

If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

This also gives us that

$$\mathbf{a} \text{ and } \mathbf{b} \text{ are } \textbf{perpendicular} \text{ (or } \textbf{orthogonal}\text{) } \Leftrightarrow \mathbf{a} \bullet \mathbf{b} = 0.$$

The direction cosines of a nonzero vector **a** are defined by

$$\cos\alpha = \frac{\mathbf{a} \bullet \vec{i}}{|\mathbf{a}|} \quad \cos\beta = \frac{\mathbf{a} \bullet \vec{j}}{|\mathbf{a}|} \quad \cos\gamma = \frac{\mathbf{a} \bullet \vec{k}}{|\mathbf{a}|}.$$

The **vector projection** of **b** onto a nonzero vector **a** is

$$\mathrm{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{a} \bullet \mathbf{a}}\,\mathbf{a}.$$

The **scalar projection** of **b** onto a nonzero vector **a** is just the length of the vector projection:

$$\mathrm{comp}_{\mathbf{a}}\,\mathbf{b} = \left\| \frac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{a} \bullet \mathbf{a}}\,\mathbf{a} \right\| = \frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\|}.$$

If $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, a_b >$ are two three-dimensional vectors, then the **cross product** of **a** and **b** = is the vector

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} =< a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 > .$$

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

If θ is the angle between **a** and **b**, then $\|\mathbf{a} \times \mathbf{b}\| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$. This is the area of the parallelogram spanned by **a** and **b**.

Also, we have **a** and **b** are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Properties of the Cross Product

$$
\begin{aligned}
&\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \\
&(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b}) \\
&\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\
&(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \\
&\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} \\
&\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}
\end{aligned}
$$

The **scalar triple product** is defined by

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

The volume of the the parallelepiped spanned by the vectors **a**, **b** and **c** is the absolute value of their scalar triple product.

A line in space is given by a point $P_0 = (x_0, y_0, z_0)$ and a nonzero vector $\mathbf{v} =< a, b, c >$. If two points P and Q are given, then **v** is the direction vector from P to Q.

The line through $P_0 = (x_0, y_0, z_0)$ and direction vector $\mathbf{v} =< a, b, c >$ has **vector equation**

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0 =< x_0, y_0, z_0 >$.

The line through $P_0 = (x_0, y_0, z_0)$ and direction vector $\mathbf{v} =< a, b, c >$ has **parametric equations**

$$
\begin{aligned}
x &= x_0 + ta \\
y &= y_0 + tb \\
z &= z_0 + tc.
\end{aligned}
$$

The line through $P_0 = (x_0, y_0, z_0)$ and direction vector $\mathbf{v} =< a, b, c >$ has **symmetric equations**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

if $abc \neq 0$. If one (or more) of the coordinates is (are) zero, this formula must be modified slightly. For example, if $a = 0$ but $bc \neq 0$, the symmetric equations are

$$\frac{y - y_0}{b} = \frac{z - z_0}{c} \text{ and } x = x_0.$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \quad 0 \leq t \leq 1.$$

A plane in space is given by a point $P_0 = (x_0, y_0, z_0)$ and a nonzero vector $\mathbf{n} =< a, b, c >$ orthogonal (or **normal**) to the plane. An equation of the plane with normal vector $\mathbf{n} =< a, b, c >$ and containing a point $P_0 = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$\mathbf{n} \bullet < x, y, z >= \mathbf{n} \bullet < x_0, y_0, z_0 > .$$

The vector $\mathbf{n}$ is a **normal vector** to the plane.

The **distance from a point** $P = (x_1, y_1, z_1)$ **to the plane with equation** $ax + by + cz = d$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

This is the scalar projection of the vector $\overrightarrow{P_0P}$ onto the normal vector $\mathbf{n}$.

Chapter 3

A **curve in the plane** is given by a vector function

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j},$$

for t in some interval $[a, b]$, where x and y are real-valued functions.

A **curve in space** is given by a vector function

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k},$$

for t in some interval $[a, b]$, where x, y and z are real-valued functions.

A curve $\vec{r}$ is **smooth** if $d\vec{r}/dt$ is continuous and never $\vec{0}$.

If $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, then

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\vec{i} + \frac{dt}{dt}\,\vec{j}.$$

$$\int \vec{r}(t)\,dt = \left(\int x(t)\,dt\right)\vec{i} + \left(\int y(t)\,dt\right)\vec{j}.$$

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\vec{i} + \left(\int_a^b y(t)\,dt\right)\vec{j}.$$

If $\vec{r}(t)$ is a curve in the plane or in space,

(1) The **velocity** is $\vec{v}(t) = \frac{d\vec{r}}{dt}$.

(2) The **speed** is $\|\vec{v}(t)\|$.

(3) The **acceleration** is $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$.

We have the following differentiation rules:

(1) $\frac{d}{dt}\vec{C} = \vec{0}$ for any constant vector-valued function $\vec{C}$.

(2) $\frac{d}{dt}\left[c\vec{u}(t)\right] = c\frac{d\vec{u}}{dt}$.

(3) $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = \frac{df}{dt}\vec{u}(t) + f(t)\frac{d\vec{u}}{dt}$.

(4) $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$.

(5) $\frac{d}{dt}\left[\vec{u}(t) \bullet \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \bullet \vec{v}(t) + \vec{u}(t) \bullet \frac{d\vec{v}}{dt}$.

(6) $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \times \vec{v}(t) + \vec{u}(t) \times \frac{d\vec{v}}{dt}$.

(7) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \frac{d\vec{u}}{dt}(f(t)) \cdot \frac{df}{dt}$.

If a projectile with position function $\vec{r}(t)$ is launched from the origin at time $t = 0$ with initial velocity $\vec{v}_0$ and $\vec{v}_0$ makes an angle α with the horizontal, then

$$\vec{r}(t) = (v_0 \cos\alpha)\, t\, \vec{i} + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right) \vec{j},$$

where $v_0 = \|\vec{v}_0\|$ and g is the acceleration due to gravity.

If $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, $a \le t \le b$, is a smooth curve then

(1) The **arclength element** is $ds = \|\vec{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

(2) The **arclength** is

$$\int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(3) The **speed** is

$$\|\vec{v}\| = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

(4) The **unit tangent vector** is

$$\vec{T}(t) = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{v}}{\|\vec{v}\|}.$$

(5) The **curvature** is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| = \frac{1}{\|\vec{v}\|} \left\| \frac{d\vec{T}}{dt} \right\|.$$

(6) The **principal unit normal vector** is

$$\vec{N}(t) = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}.$$

(7) The **unit binormal vector** is $\vec{B} = \vec{T} \times \vec{N}$.

(8) The **torsion** is

$$\tau(t) = -\frac{d\vec{B}}{ds} \bullet \vec{N}.$$

(9) If we resolve acceleration into its components parallel to $\vec{T}$ and parallel to $\vec{N}$,

$$\vec{a} = a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N}$$

we get the **tangential component** $a_{\vec{T}}$ and **normal component** $a_{\vec{N}}$ **of acceleration**:

$$a_{\vec{T}} = \frac{d^2 s}{dt^2} = \frac{d}{dt}\|\vec{v}\|$$

$$a_{\vec{N}} = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa\|\vec{v}\|^2.$$

(10) The tangential and normal components of acceleration can also be computed by

$$a_{\vec{T}} = \frac{\|\vec{v} \bullet \vec{a}\|}{\|\vec{v}\|}$$

$$a_{\vec{N}} = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}.$$

Chapter 4

If $f(x,y)$ is a function of two variables, a **level curve** of f has equation $f(x,y)=k$ for some constant k. If $f(x,y,z)$ is a function of three variables, a **level surface** of f has equation $f(x,y,z)=k$ for some constant k.

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y)=M$ then

(1) $\lim_{(x,y)\to(x_0,y_0)} f\pm g=L\pm M$.

(2) $\lim_{(x,y)\to(x_0,y_0)} fg=LM$.

(3) $\lim_{(x,y)\to(x_0,y_0)} \frac{f}{g}=\frac{L}{M}$, provided $M\neq 0$.

(4) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^{r/s}=L^{r/s}$, provided $L^{r/s}$ is defined.

A function $f(x,y)$ is **continuous** at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0).$$

If $f(x,y)$ is continuous at (x_0,y_0) and $g(t)$ is continuous at $f(x_0,y_0)$, then $g\circ f$ is continuous at (x_0,y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} g(f(x,y))=g(f(x_0,y_0)).$$

In order for the limit as (x,y) goes to (x_0,y_0) of $f(x,y)$ to exist, the limit must be independent of the path taken as (x,y) goes to (x_0,y_0). So, if you can find two different paths along which the limits as (x,y) goes to (x_0,y_0) of $f(x,y)$ are different, then the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

A continuous function $f(x,y)$ on a closed and bounded set in the plane must always has both a maximum value and a minimum value.

To find the partial derivative of f with respect to any one of its variables, treat all other variables as constants and take the derivative of f as a function of the sole remaining variable:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

The notation for higher derivatives is

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\
f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \\
f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Clairaut's Theorem says that if the second order mixed partials f_{xy} and f_{yx} are continuous at a point, then they are equal at that point.

Let $z = f(x, y)$. Let $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. The function f is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and there exist functions ϵ_1 and ϵ_2 which go to zero as Δx and Δy go to zero so that

$$\Delta z = \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.$$

What this means is that the function $z = f(x, y)$ and its tangent plane $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ are close together near the point (x_0, y_0).

Notice it is not enough for f_x and f_y to exist at a point (x_0, y_0) for f to be differentiable there. However, we do have the following theorem:

If $f(x, y)$ is a function defined on an open set containing the point (x_0, y_0), and f_x and f_y *exist* and *are continuous* at (x_0, y_0), the f is differentiable at (x_0, y_0).

The Chain Rules If $w = f(x, y)$ has continuous partial derivative f_x and f_y and if $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

If $w = f(x, y)$, $x = x(r, s)$ and $y = y(r, s)$ have continuous partial derivatives, then the composite $w = f(x(r, s), y(r, s))$ is a differentiable function of r and s and

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\
\text{and } \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.
\end{aligned}$$

These two formulas have obvious generalizations to functions of more than two variables.

If $F(x,y)$ is differentiable and the equation $F(x,y)=0$ defines y as a differentiable function of x, then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Directional Derivatives and Gradient Vectors

Let $f(x,y)$ be a function defined on an open set containing (x_0,y_0) and let $\vec{u} = u_1\,\vec{i} + u_2\,\vec{j}$ be a unit vector. The **directional derivative** of f in the direction $\vec{u}$ at the point (x_0,y_0) is

$$D_{\vec{u}}f(x_0,y_0) = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s},$$

provided this limit exists.

The **gradient vector field** of the function $f(x,y)$ is the vector field $\nabla f = f_x\,\vec{i} + f_y\,\vec{j}$.

We have the following important facts relating directional derivatives and gradient vector fields:

(1) The directional derivative of f in the direction $\vec{u}$ can be computed by the dot product

$$D_{\vec{u}}f = \nabla f \bullet \vec{u}.$$

(2) The maximum rate of increase of the function f at the point (x_0,y_0) occurs in the direction given by the gradient vector $\nabla f(x_0,y_0)$.

(3) The value of this maximum rate of increase of the function f at the point (x_0,y_0) is the length of the gradient vector $\nabla f(x_0,y_0)$:

$$\|\nabla f(x_0,y_0)\| = \sqrt{f_x(x_0,y_0)^2 + f_y(x_0,y_0)^2}$$

(4) If a curve is a level curve of the function $f(x,y)$, so that the curve is given by the equation $f(x,y)=k$ for some constant k, then the gradient vector field ∇f is perpendicular to the curve at each point. The same thing is true for level surfaces for a function of three variables.

Tangent Planes and Differentials

Let $f(x,y)$ be a differentiable function and let (x_0,y_0) be an interior point in the domain of f. Let $z_0 = f(x_0,y_0)$.

The **tangent plane** to the surface $z = f(x,y)$ at the point (x_0,y_0,z_0) is given by the equation

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Since f is differentiable, the difference $f - L$ goes to zero as (x_0, y_0) and (x, y).

Let $\Delta f = f(x,y) - f(x_0, y_0)$ be the actual change in the function between the points (x_0, y_0) and (x, y). Let $df = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ be the actual change on the tangent plane between the points (x_0, y_0) and (x, y). This is the **differential** of f at the point (x_0, y_0). Since $f - L$ goes to zero as (x_0, y_0) and (x, y), $\Delta f \approx df$.

So,

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Extreme Values and Saddle Points

If $f(x, y)$ has an extreme value at a point (x_0, y_0) and f is has first partial derivatives at (x_0, y_0), then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. A point where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ is a **critical point** for f. These are the only places (other than the boundary) where f can have extrema.

Let $f(x, y)$ have continuous second partial derivatives on an open set containing a critical point (x_0, y_0). Let $\Delta = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$ be the discriminant of f at (x_0, y_0). Then

(1) If $\Delta > 0$ and $f_{xx}(x_0, y_0) > 0$, the f has a local minimum at (x_0, y_0).

(2) If $\Delta > 0$ and $f_{xx}(x_0, y_0) < 0$, the f has a local maximum at (x_0, y_0).

(3) If $\Delta < 0$, the f has a saddle at (x_0, y_0).

This is the **second derivative test** for extrema.

Lagrange Multipliers

If we want to maximize (or minimize) a function $z = f(x, y)$ subject to a constraint $g(x, y) = 0$, the extreme value occurs when

$$\nabla f = \lambda \nabla g.$$

Similarly, if we want to maximize (or minimize) a function $w = f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$, the extreme value occurs when

$$\nabla f = \lambda \nabla g.$$

Chapter 5

If $f(x,y)$ is continuous on a rectangle $\mathscr{R} = [a,b] \times [c,d]$, then

$$\iint_{\mathscr{R}} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

So, a double integral can be computed by an iterated integral and the order of integration doesn't matter. A similar result holds for triple integrals. This is **Fubini's Theorem**.

This result is also true over more general regions as long as you compute the limits of integration correctly. So, the double integral of $f(x,y)$ over a region $\mathscr{R}$ can be computed as a (correctly set up) iterated integral with either order of integration giving the same value. A similar result holds for triple integrals.

Double (and triple) integrals have the following properties, assuming $f(x,y)$ and $g(x,y)$ are continuous:

(1) $\iint_{\mathscr{R}} cf(x,y)\,dA = c\iint_{\mathscr{R}} f(x,y)\,dA$

(2) $\iint_{\mathscr{R}} f(x,y) \pm g(x,y)\,dA = \iint_{\mathscr{R}} f(x,y)\,dA \pm \iint_{\mathscr{R}} g(x,y)\,dA$

(3) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in \mathscr{R}$, then

$$\iint_{\mathscr{R}} f(x,y)\,dA \geq \iint_{\mathscr{R}} g(x,y)\,dA.$$

(4) If $\mathscr{R}$ is the union of two nonoverlapping regions $\mathscr{R}_1$ and $\mathscr{R}_2$, then

$$\iint_{\mathscr{R}} f(x,y) = \iint_{\mathscr{R}_1} f(x,y) + \iint_{\mathscr{R}_2} f(x,y).$$

Applications

The **area** of a region $\mathscr{R}$ is given by

$$\iint_{\mathscr{R}} dA.$$

The **average value of f over a region** $\mathscr{R}$ is given by

$$\frac{1}{\text{area of } \mathscr{R}} \iint_{\mathscr{R}} f(x,y)dA.$$

The **(first) moment** of $\mathscr{R}$ with density function $\delta(x,y)$ **about the x-axis** is given by

$$M_x = \iint_{\mathscr{R}} y\,\delta(x,y)\,dA.$$

The **(first) moment** of $\mathscr{R}$ with density function $\delta(x, y)$ **about the y-axis** is given by

$$M_y = \iint_{\mathscr{R}} x\,\delta(x, y)\,dA.$$

The **center of mass** of $\mathscr{R}$ with density function $\delta(x, y)$ is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{M} \quad \text{and} \quad \overline{x} = \frac{M_x}{M}.$$

If δ is constant, the center of mass is called the **centroid** of $\mathscr{R}$.

The **moment of inertia (or second moment)** of $\mathscr{R}$ with density function $\delta(x, y)$ **about the x-axis** is given by

$$I_x = \iint_{\mathscr{R}} y^2\,\delta(x, y)\,dA.$$

The **moment of inertia (or second moment)** of $\mathscr{R}$ with density function $\delta(x, y)$ **about the y-axis** is given by

$$I_y = \iint_{\mathscr{R}} x^2\,\delta(x, y)\,dA.$$

The **polar moment of inertia** of $\mathscr{R}$ with density function $\delta(x, y)$ is given by

$$I_0 = \iint_{\mathscr{R}} (x^2 + y^2)\,\delta(x, y)\,dA.$$

Notice that $I_0 = I_x + I_y$.

The radii of gyration are

(1) About the x-axis: $R_x = \sqrt{I_x/M}$

(2) About the y-axis: $R_y = \sqrt{I_y/M}$

(3) About the origin: $R_0 = \sqrt{I_0/M}$

Double Integrals in Polar Coordinates

What you need to know here is that:

(1) $x = r\cos\theta$

(2) $y = r\sin\theta$

(3) $x^2 + y^2 = r^2$

(4) $dA = r\,dr\,d\theta$

Triple Integrals

The **volume** of a region $\mathscr{R}$ in space is given by

$$\iiint_{\mathscr{R}} dV.$$

The **average value of** $f(x,y,z)$ **over a region** $\mathscr{R}$ is given by

$$\frac{1}{\text{volume of } \mathscr{R}} \iiint_{\mathscr{R}} f(x,y,z)dV.$$

The **(first) moment of** $\mathscr{R}$ with density function $\delta(x,y,z)$ **about the** yz**-plane** is given by

$$M_{yz} = \iiint_{\mathscr{R}} x\,\delta(x,y)\,dA.$$

The **(first) moment of** $\mathscr{R}$ with density function $\delta(x,y,z)$ about the xz**-plane** is given by

$$M_{xz} = \iint_{\mathscr{R}} y\,\delta(x,y)\,dA.$$

The **(first) moment of** $\mathscr{R}$ with density function $\delta(x,y,z)$ about the xy**-plane** is given by

$$M_{xy} = \iint_{\mathscr{R}} z\,\delta(x,y)\,dA.$$

The **center of mass** of $\mathscr{R}$ with density function $\delta(x,y,z)$ is $(\overline{x},\overline{y},\overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{M} \quad \text{and} \quad \overline{y} = \frac{M_{xz}}{M} \quad \text{and} \quad \overline{z} = \frac{M_{xy}}{M}.$$

If δ is constant, the center of mass is called the centroid of $\mathscr{R}$.

The **moment of inertia (or second moment)** of $\mathscr{R}$ with density function $\delta(x,y)$ **about the** x**-axis** is given by

$$I_x = \iint_{\mathscr{R}} (y^2+z^2)\,\delta(x,y)\,dA.$$

The **moment of inertia (or second moment)** of $\mathscr{R}$ with density function $\delta(x,y)$ **about the** y**-axis** is given by

$$I_y = \iint_{\mathscr{R}} (x^2+z^2)\,\delta(x,y)\,dA.$$

The **moment of inertia (or second moment)** of $\mathscr{R}$ with density function $\delta(x,y)$ **about the** z**-axis** is given by

$$I_z = \iint_{\mathscr{R}} (x^2+y^2)\,\delta(x,y)\,dA.$$

Spherical and Cylindrical Coordinates

What you need to know here is:

In cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ dV &= r\,dr\,d\theta\,dz. \end{aligned}$$

In spherical coordinates,

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ dV &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta. \end{aligned}$$

Chapter 6

Line Integrals of Scalar Functions

If $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, $a \le t \le b$, is a curve in the plane and $\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a vector field, we compute

$$\vec{v} = x'(t)\,\vec{i} + y'(t)\,\vec{j},$$

and

$$\vec{T} = \frac{x'(t)\,\vec{i} + y'(t)\,\vec{j}}{\|x'(t)\,\vec{i} + y'(t)\,\vec{j}\|} = \frac{x'(t)\,\vec{i} + y'(t)\,\vec{j}}{ds/dt}.$$

Then

$$\begin{aligned}\int_C \vec{F} \bullet \vec{T}\,ds &= \int_a^b \Big(P(x(t),y(t))\,\vec{i} + Q(x(t),y(t))\,\vec{j}\Big) \bullet \frac{x'(t)\,\vec{i} + y'(t)\,\vec{j}}{ds/dt} \cdot \frac{ds}{dt}\,dt \\ &= \int_a^b \Big(P(x(t),y(t))\,\vec{i} + Q(x(t),y(t))\,\vec{j}\Big) \bullet \Big(x'(t)\,\vec{i} + y'(t)\,\vec{j}\Big)\,dt \\ &= \int_a^b \big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\big)\,dt \\ &= \int_C P(x,y)\,dx + Q(x,y)\,dy.\end{aligned}$$

Similarly, if $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$, $a \le t \le b$, is a curve in space and $\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + R(x,y)\,\vec{k}$ is a vector field, then

$$\int_C \vec{F} \bullet \vec{T}\,ds = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P(x(t),y(t))\,\frac{dx}{dt} + Q(x(t),y(t))\,\frac{dy}{dt} + R(x(t),y(t))\,\frac{dz}{dt}\right)dt.$$

Work, Flow, Circulation, and Flux

The integral

$$\int_C \vec{F} \bullet \vec{T}\,ds$$

is the **flow** along C.

If the curve C is closed, the integral

$$\oint_C \vec{F} \bullet \vec{T}\,ds$$

is the **circulation** along C.

If the curve C is closed, the integral

$$\oint_C \vec{F} \bullet \vec{n}\,ds$$

is the **flux** along C, and if C is a plane curve and $\vec{F} = P\,\vec{i} + Q\,\vec{j}$, then flux is given by

$$\int_C \vec{F} \bullet \vec{n}\, ds = \int_C -Q\, dx + P\, dy.$$

The integral

$$\int_C \vec{F} \bullet \vec{T}\, ds$$

is also the **work** done in moving an object along the curve in the positive direction. All the following are different forms of the same expression:

$$\begin{aligned} \vec{W} &= \int_C \vec{F} \bullet \vec{T}\, ds \\ &= \int_C \vec{F} \bullet d\vec{r} \\ &= \int_a^b \vec{F} \bullet \frac{d\vec{r}}{dt}\, dt \\ &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right)\, dt \\ &= \int_C P\, dx + Q\, dy + R\, dz. \end{aligned}$$

If a thin wire in the shape of a curve C has density function δ, then the **mass** of the wire is given by

$$M = \int_C \delta\, ds,$$

the **moment** of the wire **about the** yz**-plane** is given by

$$M_{yz} = \int_C x\, \delta\, ds,$$

the **moment** of the wire **about the** xz**-plane** is given by

$$M_{xz} = \int_C y\, \delta\, ds,$$

the **moment** of the wire **about the** xy**-plane** is given by

$$M_{xy} = \int_C z\, \delta\, ds,$$

and the **center of mass** of the wire is $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{M}, \quad \overline{y} = \frac{M_{xz}}{M}, \quad \overline{z} = \frac{M_{xy}}{M}.$$

Path Independence, Conservative Fields, and Potential Functions

A vector field $\vec{F}$ is **conservative** if $\vec{F} = \nabla f$ for some function f. So, in the plane, this means

$$\vec{F} = P\,\vec{i} + Q\,\vec{j} = f_x\,\vec{i} + f_y\,\vec{j} = \nabla f,$$

so

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}.$$

The function f is a **potential function** for $\vec{F}$.

If $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ has continuous first order partial derivatives and if $\vec{F}$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

This is a consequence of Clairaut's Theorem.

If the vector field $\vec{F}$, in addition, is defined on a simply connected region, then the converse is also true:

If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\vec{F}$ is conservative. This is a consequence of Green's Theorem.

If $\vec{F}$ is conservative with potential function f and C is any path from a point A to a point B, then

$$\int_C \vec{F} \bullet \vec{T}\, ds = f(B) - f(A).$$

This is the Fundamental Theorem of Line Integrals. In particular, if $\vec{F}$ is conservative, the integral is independent of path.

Conversely, if $\vec{F}$ is defined on a simply connected region D and the integral $\int_C \vec{F} \bullet \vec{T}\, ds$ is independent of path for every curve C in D, then $\vec{F}$ is conservative.

If $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ is conservative, then $\operatorname{curl} \vec{F} = \vec{0}$. If $\vec{F}$ is defined on a simply connected domain D and $\operatorname{curl} \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Surface Integrals

Let S be a smooth surface in space parametrized by $\vec{r}(u, v) = x(u, v)\,\vec{i} + y(u, v)\,\vec{j} + z(u, v)\,\vec{k}$ for (u, v) in some region $\mathscr{R}$ in the uv-plane. Then

$$dS = \left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\| dA$$

and

$$\int_S f(x, y, z)\, dS = \int_{\mathscr{R}} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right\| dA.$$

The length of $\vec{r}_u \times \vec{r}_v$ is the "stretch-squish" (technically, the Jacobian) and measures how the parametrization between the uv-plane and xyz-space distorts area.

If $z = f(x, y)$, so that the surface S is parametrized by a region $\mathscr{R}$ in the xy-plane, then

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA.$$

If S is a surface and $\vec{n}$ is a unit normal vector to S, then the integral

$$\int_S \vec{F} \bullet \vec{n}\, dS$$

is the flux of the vector field $\vec{F}$ across the surface S in the direction $\vec{n}$.

If a thin shell has the shape of a surface S with density function $\delta(x, y, z)$, the **mass of the shell** is

$$\iint_S \delta\, dS.$$

The **moment of the shell about the yz-plane** is given by

$$\iint_S x\, \delta\, dS.$$

The **moment of the shell about the xz-plane** is given by

$$\iint_S y\, \delta\, dS.$$

The **moment of the shell about the xy-plane** is given by

$$\iint_S z\, \delta\, dS.$$

The **center of mass of the shell** is given by $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{M}, \quad \overline{y} = \frac{M_{xz}}{M}, \quad \overline{z} = \frac{M_{xy}}{M}.$$

Vector Calculus

If f is a function with first order partial derivatives, then the **gradient** of f is the vector field

$$\nabla f = f_x\, \vec{i} + f_y\, \vec{j}\, (+ f_z\, \vec{k}).$$

If $\vec{F} = P(x, y, z)\, \vec{i} + Q(x, y, z)\, \vec{j} + R(x, y, z)\, \vec{k}$ has first order partial derivatives, then the **curl** of $\vec{F}$ is the vector field

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial\ /\partial x & \partial\ /\partial y & \partial\ /\partial z \\ P & Q & R \end{pmatrix}.$$

If $\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ has first order partial derivatives, then the **divergence** of $\vec{F}$ is the scalar function

$$\operatorname{div}\vec{F} = \nabla \bullet \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Handy Dandy Facts

(1) $\operatorname{curl}(\nabla f) = \vec{0}$ if f has continuous second order partial derivatives. So, if $\vec{F}$ is conservative and has continuous first order partial derivatives, then $\operatorname{curl}(\vec{F}) = \vec{0}$.

(2) $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ if $\vec{F}$ has continuous second order partial derivatives.

Conversely, if everything is defined over a simply connected region D, then the converses of these statements are true:

(1) If $\operatorname{curl}(\vec{F}) = \vec{0}$ and $\vec{F}$ has continuous first order partial derivatives, then $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$ for some f.

(2) If $\operatorname{div}(\vec{G}) = 0$ and $\vec{G}$ has continuous first order partial derivatives, with then there is a vector field $\vec{F}$ so that $\vec{G} = \operatorname{curl}\vec{F}$.

Green's Theorem

If C is a piecewise smooth simple closed curve oriented counterclockwise and enclosing a region $\mathscr{R}$ in the plane, and $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ is a vector field with P and Q having continuous first order partial derivatives, then

$$\oint_C \vec{F} \bullet \vec{T}\,ds = \oint_C P\,dx + Q\,dy = \iint_{\mathscr{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

The integral on the left is the **circulation** of $\vec{F}$ around C.

Stokes' Theorem

Let S is a piecewise smooth oriented surface having a piecewise smooth boundary curve C oriented postively with respect to the orientation of S. Let $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then

$$\oint_C \vec{F} \bullet \vec{T}\,ds = \iint_S \operatorname{curl}\vec{F} \bullet \vec{n}\,dS.$$

The integral on the left is the **circulation** of $\vec{F}$ around C.

The Divergence Theorem

Let $\vec{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface bounding a region D in space. Then

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV.$$

The integral on the left is the **flux** of $\vec{F}$ across S.