Substitution

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Outline

Indefinite Integrals and the Substitution Method

Indefinite Integrals and the Substitution Method

Let's recall the Chain Rule:

If F is a differentiable function of u and u is a differentiable function of x, then F is a differentiable function of x and

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx}.$$

Indefinite Integrals and the Substitution Method

Now, suppose f is a continuous function of u and F is an antiderivative of f with respect to u.

Then we have

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = f(u)\,\frac{du}{dx}.$$

Indefinite Integrals and the Substitution Method

If we reverse this process by integration, we get

$$F(u(x)) = \int \frac{dF}{dx}\, dx = \int f(u)\, \frac{du}{dx}\, dx.$$

If we abuse the notation and write $\frac{du}{dx}\, dx$ as du, we get

$$F(u(x)) = \int f(u)\, du,$$

where $dF/du = f$.

Indefinite Integrals and the Substitution Method

Integration by u-substitution

$$\int f(u(x))\,\frac{du}{dx}\,dx = \int f(u)\,du = F(u(x)) + C,$$

where $dF/du = f$.

The Process of u-Substitution

Indefinite Integrals and the Substitution Method

This process is called **integration by u-substitution**. Here's how you do it:

1. Find something in the integrand whose derivative is also in the integrand.
2. Let u be that something.
3. Compute $\frac{du}{dx}$ and abuse the notation.
4. Substitute u's and du into the integral in place of the x's and dx.
5. Find the antiderivative.
6. Resubstitute to get the original antiderivative.

Examples

Example 1

Example

Evaluate the antiderivative

$$\int 2x\sqrt{x^2+4}\,dx.$$

Example 1

Solution

Notice that the derivative of $x^2 + 4$ *is* $2x$*. So, we let* $u = x^2 + 4$*. Then, we take the derivative and abuse the notation.*

$$\frac{du}{dx} = 2x$$
$$du = 2x\,dx.$$

Example 1

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int 2x\sqrt{x^2+4}\,dx &= \int \sqrt{x^2+4}\cdot 2x\,dx \\ &= \int \sqrt{u}\,du \\ &= \frac{2}{3}u^{3/2}+C \\ &= \frac{2}{3}(x^2+4)^{3/2}+C.\end{aligned}$$

Example 2

Example

Evaluate the antiderivative

$$\int x \sin(2x^2)\, dx$$

Example 2

Solution

Notice that the derivative of $2x^2$ *is* $4x$*. We can fudge the 4 as long as we have the* x*. So, we let* $u = 2x^2$*. Then, we take the derivative and abuse the notation.*

$$\frac{du}{dx} = 4x, \quad du = 4x\,dx.$$

Now divide both sides by 4:

$$\frac{1}{4}\,du = x\,dx.$$

Example 2

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}
\int x \sin(2x^2)\, dx &= \int \sin(2x^2) \cdot x\, dx \\
&= \int \sin(u) \cdot \frac{1}{4}\, du \\
&= \frac{1}{4} \int \sin(u)\, du \\
&= -\frac{1}{4} \cos(u) + C \\
&= -\frac{1}{4} \cos(2x^2) + C.
\end{aligned}$$

Example 3

Example

Evaluate the antiderivative

$$\int \frac{9x^2\,dx}{\sqrt{1-x^3}}.$$

Example 3

Solution

Notice that the derivative of $1 - x^3$ *is* $-3x^2$*. We can fudge the* -3 *as long as we have the* x^2*. So, we let* $u = 1 - x^3$*. Then, we take the derivative and abuse the notation.*

$$\frac{du}{dx} = -3x^2, \quad du = -3x^2\,dx.$$

Now multiply both sides by -3*.*

$$-3\,du = 9x^2\,dx.$$

Example 3

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int \frac{9x^2\,dx}{\sqrt{1-x^3}} &= \int \frac{-3\,du}{\sqrt{u}} \\ &= -3\int u^{-1/2}\,du \\ &= -3\cdot 2u^{1/2}+C \\ &= -6(1-x^3)^{1/2}+C \\ &= -6\sqrt{1-x^3}+C.\end{aligned}$$

Example 4

Example

Evaluate the antiderivative

$$\int \tan^2 x \sec^2 x \, dx.$$

Example 4

Solution

Notice that the derivative of $\tan x$ *is* $\sec^2 x$. *So, we let* $u = \tan x$. *Then, we take the derivative and abuse the notation.*

$$\frac{du}{dx} = \sec^2 x$$
$$du = \sec^2 x \, dx.$$

Example 4

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int \tan^2 x \sec^2 x\, dx &= \int (\tan x)^2 \cdot \sec^2 x\, dx \\ &= \int u^2\, du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\tan^3 x + C.\end{aligned}$$

Example 5

Example

Evaluate the antiderivative

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx.$$

Example 5

Solution

Notice that the derivative of $1+\sqrt{x} = 1 + x^{1/2}$ *is* $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. *We can fudge the 2. So, we let* $u = 1+\sqrt{x}$. *Then, we take the derivative and abuse the notation.*

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \qquad du = \frac{1}{2\sqrt{x}}\,dx.$$

Now, multiply both sides by 2:

$$2\,du = \frac{1}{\sqrt{x}}\,dx.$$

Example 5

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx &= \int (1+\sqrt{x})^{1/3}\cdot\frac{1}{\sqrt{x}}\,dx\\ &= \int u^{1/3}\cdot 2\,du\\ &= 2\int u^{1/3}\,du\\ &= 2\cdot\frac{3}{4}u^{4/3}+C\\ &= \frac{3}{2}(1+\sqrt{x})^{4/3}+C.\end{aligned}$$

Example 6

Example

Evaluate the antiderivative

$$\int \frac{dx}{1+4x^2}.$$

Example 6

Solution

Notice that $\frac{1}{1+4x^2} = \frac{1}{1+(2x)^2}$ *and the derivative of* $2x$ *is 2, which we can fudge. So, we let* $u = 2x$. *Then, we take the derivative and abuse the notation.*

$$\frac{du}{dx} = 2, \qquad du = 2\,dx.$$

Now, multiply both sides by $\frac{1}{2}$*:*

$$\frac{1}{2}\,du = dx.$$

Example 6

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}
\int \frac{dx}{1+4x^2} &= \int \frac{1}{1+(2x)^2}\,dx \\
&= \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du \\
&= \frac{1}{2}\int \frac{1}{1+u^2}\,du \\
&= \frac{1}{2}\arctan(u)+C \\
&= \frac{1}{2}\arctan(2x)+C.
\end{aligned}$$

Example 6

Important Comment

Notice this means you need to be able to recognize what expression is a derivative.

That means you have to know all the derivatives of the basic functions!

Example 7

Example

Evaluate the antiderivative

$$\int \frac{dx}{\sqrt{5x+8}}.$$

Example 7

Solution

We are going to work this problem two different ways to show you there is not just one path to the solution.

Example 7

Solution

Let $u = 5x + 8$. *Then* $\frac{du}{dx} = 5$, *so* $du = 5\,dx$ *and* $\frac{1}{5}\,du = dx$.

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned} \int \frac{dx}{\sqrt{5x+8}} &= \int \frac{1}{\sqrt{5x+8}}\,dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{5}\,du \\ &= \frac{1}{5}\int u^{-1/2}\,du = \frac{1}{5} \cdot 2u^{1/2} + C \\ &= \frac{2}{5}\sqrt{5x+8} + C. \end{aligned}$$

Example 7

Solution

On the other hand, we can let $u = \sqrt{5x+8}$. *Now,* **square** *both sides to get* $u^2 = 5x + 8$. *Now compute the derivative implicitly and abuse the notation.*

$$\begin{aligned} 2u\,\frac{du}{dx} &= 5 \\ 2u\,du &= 5\,dx. \end{aligned}$$

Now, divide by 5.

$$\frac{2}{5}u\,du = dx.$$

Example 7

Solution

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{\sqrt{5x+8}}\,dx &= \int \frac{1}{u}\cdot\frac{2}{5}u\,du\\ &= \frac{2}{5}\int 1\,du = \frac{2}{5}u + C\\ &= \frac{2}{5}\sqrt{5x+8} + C.\end{aligned}$$

Notice you get the same answer either way.

Example 8

Example

Evaluate the antiderivative

$$\int \sin x \cos x \, dx.$$

Example 8

Solution

We are going to work this problem two different ways and get two different answers!

Example 8

Solution

Let $u = \sin x$. *Then* $\frac{du}{dx} = \cos x$, *so* $du = \cos x\,dx$.

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int \sin x \cos x\,dx &= \int u\,du \\ &= \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.\end{aligned}$$

Example 8

Solution

On the other hand, we can let $u = \cos x$*, so* $du = -\sin x\, dx$*. Multiply by* -1 *to get* $-du = \sin x\, dx$*.*

Now, substitute, evaluate the antiderivative, and resubstitute.

$$\begin{aligned}\int \sin x \cos x\, dx &= \int \cos x \sin x\, dx \\ &= \int u\,(-du) = -\int u\, du \\ &= -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C.\end{aligned}$$

Example 8

Solution

Why did we get two answers: $\frac{1}{2}\sin^2 x$ *and* $-\frac{1}{2}\cos^2 x$*?*

Notice that

$$\frac{1}{2}\sin^2 x - \left(-\frac{1}{2}\cos^2 x\right) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2},$$

so these two answers differ by a constant—just as they must.

So, both answers are correct.

Substitution in Definite Integrals

Substitution in Definite Integrals

Fortunately, we have done almost all the work we have to do here in the last section.

When you make a u-substitution in a definite integral, just as in the last section, you substitute u's for x's, du's for dx's, **and** you have to change the **limits**.

Examples

Example 1

Example

Evaluate the definite integral

$$\int_0^{\pi} 3\cos^2 x \sin x \, dx.$$

Example 1

Solution

Let $u = \cos x$. *Then* $du = -\sin x\, dx$ *and* $-du = \sin\ dx$. *This integral goes from* $x = 0$ *to* $x = \pi$. *When* $x = 0$, $u = 1$. *When* $x = \pi$, $u = -1$. *Substituting, we get*

$$\begin{aligned} \int_0^{\pi} 3\cos^2 x \sin x\, dx &= \int_1^{-1} -3u^2\, du \\ &= -u^3\Big|_1^{-1} \\ &= -(-1)^3 - (-(1)^3) = 2. \end{aligned}$$

Example 2

Example

Evaluate the definite integral

$$\int_{1}^{9} \sqrt{4+5x}\, dx.$$

Example 2

Solution

Let $u = 4 + 5x$*. Then* $du = 5\,dx$ *and* $\frac{1}{5}du = dx$*. When* $x = 1$*,* $u = 9$*. When* $x = 9$*,* $u = 49$*. Substituting, we get*

$$\begin{aligned}\int_1^9 \sqrt{4+5x}\,dx &= \int_9^{49} \sqrt{u} \cdot \frac{1}{5}\,du \\ &= \frac{1}{5}\int_9^{49} \sqrt{u}\,du = \frac{1}{5}\left[\frac{2}{3}u^{3/2}\right|_9^{49} \\ &= \frac{2}{15}\left(49^{3/2} - 9^{3/2}\right) = \frac{632}{15}.\end{aligned}$$