The Fundamental Theorem of Calculus

William M. Faucette

University of West Georgia

Outline

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals

Let $f(x)$ be a continuous function defined on a closed interval $[a, b]$. Then there is some point c, $a < c < b$, so that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The Mean Value Theorem for Integrals

Let's see why this is true.

Since f is a continuous function on a closed interval, f has a maximum value M and a minimum value m on $[a, b]$. So, we have

$$m \leq f(x) \leq M \qquad \text{for all } x \in [a, b].$$

Then we have

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x) \leq \int_a^b M\,dx = M(b-a).$$

The Mean Value Theorem for Integrals

In the equation

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x) \leq \int_a^b M\,dx = M(b-a),$$

we evaluate the integral on the left by knowing what the integral means: It's the area under the line $y = m$ from a to b. This region is a rectangle of height m and width $b-a$, so the area is $m(b-a)$.

The same is true for the integral on the right.

The Mean Value Theorem for Integrals

Now divide by $(b - a)$:

$$m \leq \frac{1}{b-a} \int_a^b f(x) \leq M.$$

The Mean Value Theorem for Integrals

Recall the Intermediate Value Theorem:

Intermediate Value Theorem

Let f be a continuous function on the interval $[a, b]$. If $f(x_1) = y_1$ and $f(x_2) = y_2$, and y is any number between y_1 and y_2, then there exists x, $x_1 \leq x \leq x_2$ so that $f(x) = y$.

This theorem says if f takes on one value and then a second value, then f must take on every intermediate value on that interval.

The Mean Value Theorem for Integrals

We have found that

$$m \leq \frac{1}{b-a}\int_a^b f(x) \leq M.$$

Since a continuous function actually takes on its minimum and maximum values, by the Intermediate Value Theorem, there exists c in the interval $[a, b]$ so that

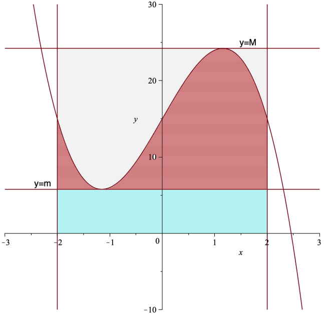
$$f(c) = \frac{1}{b-a}\int_a^b f(x).$$

This is the Mean Value Theorem for Integrals.

The Mean Value Theorem for Integrals

Here's a sketch which explains the Mean Value Theorem for Integrals.

Figure: Sketch for Mean Value Theorem for Integrals



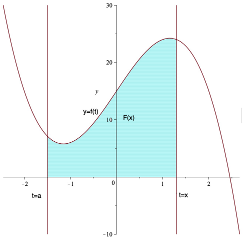
The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus, Part I

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$.

Figure: Sketch of $F(x)$



The Fundamental Theorem of Calculus, Part I

Proof.

By definition,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

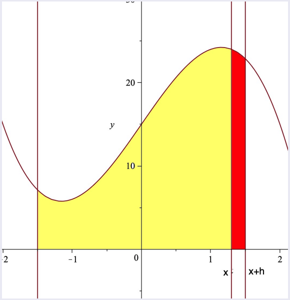
See the sketch on the next slide.

□

The Fundamental Theorem of Calculus, Part I

Proof.

Figure: Sketch of $F(x+h) - F(x)$



The Fundamental Theorem of Calculus, Part I

Proof.

By the Mean Value Theorem for Integrals, there exists c_h lying between x and $x+h$, depending on h, so that

$$f(c_h) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

The Fundamental Theorem of Calculus, Part I

Proof (cont.)

So,

$$
\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \\
&= \lim_{h \to 0} f(c_h) \\
&= f(x).
\end{aligned}
$$

Since c_h is between x and $x + h$, as h goes to 0, c_h goes to x. By continuity, $f(c_h)$ goes to $f(x)$. $\square$

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus, Part I, tells you something very important.

It tells you that every continuous function has an antiderivative.

Examples

Example 1

Example

Define

$$F(x) = \int_0^x e^{t^2}\,dt.$$

By the Fundamental Theorem of Calculus, Part I, $F(x)$ is differentiable everywhere and $F'(x) = e^{x^2}$.

Example 2

Example

Define
$$F(x) = \int_0^{3x} sin(t)\,dt.$$
Let $u = 3x$. Then
$$F(u) = \int_0^{u} \sin(t)\,dt.$$
By the Fundamental Theorem of Calculus, Part I, $F(u)$ is differentiable everywhere and $F'(u) = \sin(u)$.
By the Chain Rule, $F(x)$ is differentiable everywhere and
$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = \sin(u) \cdot 3 = 3\sin(3x).$$

Fundamental Theorem of Calculus, Part 2

Fundamental Theorem of Calculus, Part 2

Theorem

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This means if the integrand f is continuous on $[a, b]$ and you want to evaluate $\int_a^b f(x)\,dx$, all you have to do is to find any antiderivative F of f, evaluate $F(b)$, evaluate $F(a)$, and subtract.

Fundamental Theorem of Calculus, Part 2

Proof.

Let f be continuous on the interval $[a, b]$ and let F be any antiderivative of f on $[a, b]$.

By the Fundamental Theorem of Calculus, Part I,

$$G(x) = \int_a^x f(t)\,dt$$

is also an antiderivative of f.

Fundamental Theorem of Calculus, Part 2

Proof (cont.)

One corollary of the Mean Value Theorem tells us that any two antiderivatives of a continuous function must differ by a constant. So, $G(x) = F(x) + C$ for some constant C.

If we evaluate $G(a)$, we get $G(a) = \int_a^a f(t)\,dt = 0$, so $0 = G(a) = F(a) + C$. This tells use that $C = -F(a)$.

Then $G(x) = \int_a^x f(t)\,dt = F(x) - F(a)$. In particular,

$$G(b) = \int_a^b f(t)\,dt = F(b) - F(a).$$

This concludes the proof. $\square$

Examples

Example 3

Example

Evaluate

$$\int_0^{\pi} (1 + \cos x)\, dx$$

Example 3

Solution

First, we find an antiderivative of $1 + \cos x$.

$$\int (1 + \cos x)\, dx = x + \sin x + C.$$

By the Fundamental Theorem of Calculus, Part 2,

$$\begin{aligned} \int_0^{\pi} (1 + \cos x)\, dx &= \left. x + \sin x \right|_0^{\pi} \\ &= (\pi + \sin \pi) - (0 + \sin 0) = \pi. \end{aligned}$$

Example 4

Example

Evaluate the integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} (x+1)(x^2+4)\,dx$$

Example 4

Solution

$$\int_{-\sqrt{3}}^{\sqrt{3}} (x+1)(x^2+4)\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} (x^3+x^2+4x+4)\,dx$$

$$= \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + 2x^2 + 4x\right)\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \left(\frac{1}{4}(\sqrt{3})^4 + \frac{1}{3}(\sqrt{3})^3 + 2(\sqrt{3})^2 + 4(\sqrt{3})\right)$$

$$- \left(\frac{1}{4}(-\sqrt{3})^4 + \frac{1}{3}(-\sqrt{3})^3 + 2(-\sqrt{3})^2 + 4(-\sqrt{3})\right)$$

$$= 10\sqrt{3}.$$

Example 5

Example

Evaluate

$$\int_0^{\pi/6} (\sec x + \tan x)^2 \; dx$$

Example 5

Solution

First, we do a little precalculus.

$$\begin{aligned}(\sec x + \tan x)^2 &= \sec^2 x + 2\sec x \tan x + \tan^2 x \\ &= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1) \\ &= 2\sec^2 x + 2\sec x \tan x - 1.\end{aligned}$$

Example 5

Solution

By the Fundamental Theorem of Calculus, Part 2,

$$\begin{aligned}
\int_0^{\pi/6} (\sec x + \tan x)^2 \; dx &= \int_0^{\pi/6} 2\sec^2 x + 2\sec x \tan x - 1 \, dx \\
&= \left(2\tan x + 2\sec x - x\right)\Big|_0^{\pi/6} \\
&= \left(2\tan\left(\frac{\pi}{6}\right) + 2\sec\left(\frac{\pi}{6}\right) - \left(\frac{\pi}{6}\right)\right) \\
&\quad - \left(2\tan 0 + 2\sec 0 - 0\right) \\
&= 2 \cdot \frac{\sqrt{3}}{3} + 2 \cdot \frac{2\sqrt{3}}{3} - \frac{\pi}{6} - 2 \\
&= 2\sqrt{3} - \frac{\pi}{6} - 2.
\end{aligned}$$

Example 6

We have a dash of déjà vu:

Example

Evaluate the integral

$$\int_a^b mx\, dx$$

Example 6

Solution

We use the Fundamental Theorem of Calculus, Part 2:

$$\begin{aligned}\int_a^b mx\,dx &= \frac{1}{2}mx^2\bigg|_a^b \\ &= \frac{1}{2}mb^2 - \frac{1}{2}ma^2.\end{aligned}$$