Approximating Areas

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Outline

Finite Sums and Sigma Notation

Finite Sums and Sigma Notation

If we are going to be adding up more and more terms, we need some notation to deal with this. The notation is **sigma notation** and it looks like this:

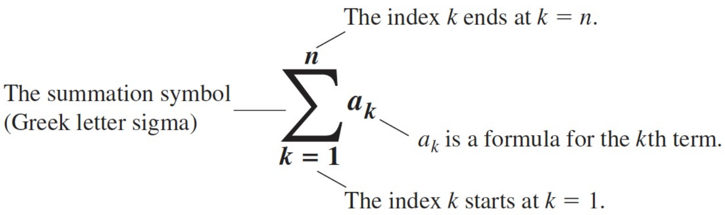
$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The sum on the right is the sum represented by the sigma notation on the left.

Finite Sums and Sigma Notation

The letter k is the **index** of the sum.

Figure: Anatomy of a Summation



Examples of Sigma Notation

Examples of Sigma Notation

$$\sum_{k=1}^{8} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$

$$\sum_{k=1}^{4} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$\sum_{k=1}^{5} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 62$$

Algebra Rules for Finite Sums

Algebra Rules for Finite Sums

1. Sum Rule: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. Difference Rule: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. Constant Multiple Rule: $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
4. Constant Value Rule: $\sum_{k=1}^{n} c = n \cdot c$

An Important Formula

An Important Formula

Example

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

An Important Formula

Proof

Let

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

Now reverse the order of the summands in S:

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

An Important Formula

Proof.

We have

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n,$$
$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Now, add the two sums from top to bottom

$$\begin{aligned} 2S &= \overbrace{(n+1) + (n+1) + \cdots + (n+1) + (n+1)}^{n \text{ times}} \\ &= n(n+1). \end{aligned}$$

Dividing the equation by 2 gives the result. □

More Important Formulas

More Important Formulas

The first n squares: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes: $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Limits of Finite Sums

Limits of Finite Sums

Start with the function $f(x) = x^2$ on the interval $[0, 1]$.

We divide the interval into n pieces of equal width. These points are

$$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1.$$

The set of these points is called a **partition** of the interval $[0, 1]$.

For notation, $x_0 = 0$, $x_1 = 1/n$, $x_2 = 2/n$, etc. In general, we have $x_k = k/n$.

Limits of Finite Sums

We now need to choose a point in the kth interval where we will evaluate the function to get the height of the rectangle.

It doesn't matter which point you pick, so we'll pick the right endpoint, k/n. The height of the kth rectangle is then $f(k/n)$.

The area of the kth rectangle is then $f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \frac{k^2}{n^3}$.

The sum of the areas of these n rectangles is then

$$\sum_{k=1}^{n} \frac{k^2}{n^3}.$$

Limits of Finite Sums

We'll use our list of important formulas and the properties of sums to write this sum in **closed form**.

$$\begin{aligned}\sum_{k=1}^{n} \frac{k^2}{n^3} &= \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\ &= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6n^2}.\end{aligned}$$

Limits of Finite Sums

Since the estimate of the area should get better as we divide the interval into more and more pieces, we take the limit as n goes to infinity.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} &= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \\
&= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \cdot \frac{1/n^2}{1/n^2} \\
&= \lim_{n\to\infty} \frac{2+3\frac{1}{n}+\frac{1}{n^2}}{6} \\
&= \frac{2+3(0)+(0)}{6} \\
&= \frac{1}{3}.
\end{aligned}$$

This is the area under the graph of $y = x^2$ over the interval $[0, 1]$.

Riemann Sums

Riemann Sums

Start with a bounded function $f(x)$ defined on an interval $[a, b]$.

We choose a **partition** $P = \{x_1, x_2, \dots, x_{n-1}, x_n\}$ between a and b so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

The partition P divides the interval $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

The kth subinterval is $[x_{k-1}, x_k]$. We denote its width by $\Delta x_k = x_k - x_{k-1}$.

The largest value of Δx_k is the **mesh** or **norm** of the partition. It is denoted $\|P\|$.

Riemann Sums

In the interval $[x_{k-1}, x_k]$, we choose a point x_k^*. This is the point where we will evaluate f to get the height of the kth rectangle. So, the height of the kth rectangle is $f(x_k^*)$.

The area of the kth rectangle is $f(x_k^*)\,\Delta x_k$, and the sum of the areas of the n rectangles is

$$S_P = \sum_{k=1}^{n} f(x_k^*)\Delta x_k,$$

which is the **Riemann sum for f on the interval $[a, b]$ with respect to the partition P.**

This man's name is pronounced "Rē′· mŏn".

Riemann Sums

If all the subintervals have the same width, the partition is called **regular**. In this case, we denote the width of each rectangle is given by

$$\Delta x = \frac{b-a}{n}.$$

Also, in the case of a regular partition, we have

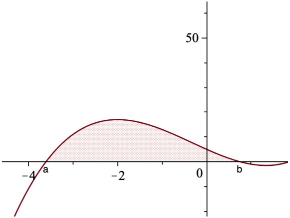
$$x_k = a + k\,\Delta x = a + k\,\frac{b-a}{n}$$

Area and Estimating with Finite Sums

Area and Estimating with Finite Sums

Suppose we have a function $f(x)$ which is non-negative on the interval $[a, b]$. How do we find the area under the graph $y = f(x)$ over the interval $[a, b]$?

Figure: Graph of $y = f(x)$ on $[a, b]$



Area and Estimating with Finite Sums

We look at the graph $y = 1 - x^2$ on the interval $[0, 1]$.

First, divide the interval $[0, 1]$ into two equal subintervals $[0, 0.5]$ and $[0.5, 1]$.

One each subinterval, we will use the left endpoint to establish the height of a rectangle over the subinterval by finding the value of the function there.

Area and Estimating with Finite Sums

The first left endpoint, $x = 0$, gives us a value of $y(0) = 1$. We construct a rectangle with height 1 over the interval $[0, 0.5]$. We what we have done is to assume the function has the constant value 1 on this small interval.

The second left endpoint, $x = 0.5$, gives us a value of $y(0.5) = 3/4$. We construct a rectangle with height $3/4$ over the interval $[0.5, 1]$. We what we have done is to assume the function has the constant value $3/4$ on this small interval.

Area and Estimating with Finite Sums

The sum of the areas of the two rectangles is

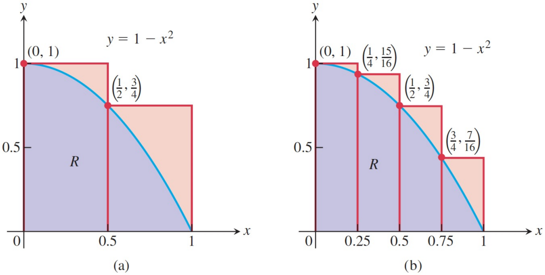
$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{3}{4} = \frac{7}{8}.$$

This is an approximation for the area under the curve $y = 1 - x^2$ over the interval $[0, 1]$.

Area and Estimating with Finite Sums

To get a better estimate of the area, we divide the interval $[0, 1]$ into four subintervals: $[0, 1/4]$, $[1/4, 1/2]$,$[1/2, 3/4]$, and $[3/4, 1]$.

Figure: Approximating Area



Area and Estimating with Finite Sums

The first left endpoint, $x = 0$, gives us a value of $y(0) = 1$. We construct a rectangle with height 1 over the interval $[0, 1/4]$. We what we have done is to assume the function has the constant value 1 on this small interval.

The second left endpoint, $x = 1/4$, gives us a value of $y(1/4) = 15/16$. We construct a rectangle with height $15/16$ over the interval $[1/4, 1/2]$. We what we have done is to assume the function has the constant value $15/16$ on this small interval.

The third left endpoint, $x = 1/2$, gives us a value of $y(1/2) = 3/4$. We construct a rectangle with height $3/4$ over the interval $[1/2, 3/4]$. We what we have done is to assume the function has the constant value $3/4$ on this small interval.

Area and Estimating with Finite Sums

The fourth left endpoint, $x = 3/4$, gives us a value of $y(3/4) = 7/16$. We construct a rectangle with height $3/4$ over the interval $[0.5, 1]$. We what we have done is to assume the function has the constant value $7/16$ on this small interval.

The sum of the areas of the four rectangles is

$$\frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{15}{16} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{7}{16} = \frac{25}{32}.$$

This is a better approximation for the area under the curve $y = 1 - x^2$ over the interval $[0, 1]$.

Area and Estimating with Finite Sums

If you use the point in each subinterval on which the function has its maximum on that subinterval, then the sum is called an **upper sum**. This estimate is larger than (or equal to) the area under the curve.

If you use the point in each subinterval on which the function has its minimum on that subinterval, then the sum is called a **lower sum**. This estimate is smaller than (or equal to) the area under the curve.

The area under the curve is somewhere between the lower sum and the upper sum.

Area and Estimating with Finite Sums

You can actually choose any point in each subinterval. Most customarily, one uses the left endpoint, the right endpoint, or the midpoint of the subinterval. You will (in general) get different estimates for the area under the curve.

Area and Estimating with Finite Sums

How do you get better and better estimates? You divide the interval into more and more subintervals.

Figure: Table of Lower Sums, Midpoint Sums, and Upper Sums

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

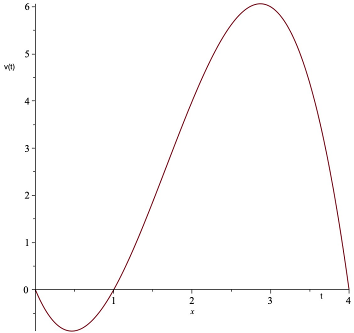
In particular, it appears the area under the curve $y = 1 - x^2$ over the interval $[0, 1]$ is $2/3$.

Distance Traveled

Distance Traveled

Suppose we're given the velocity function $v(t)$ for a particle moving along an axis. If we graph this function, we get the graph $y = v(t)$.

Figure: Graph $y = v(t)$



Distance Traveled

If we divide the interval $[1, 4]$ into subintervals, the width of each subintervals if Δt, a change in time, and the height of the approximating rectangle is a velocity. The product of time and velocity is distance.

This interprets the area under the graph $y = v(t)$ as the **distance traveled** between $t = 1$ and $t = 4$.

Displacement versus Distance Traveled

If we divide the interval $[0, 1]$ into subintervals, the width of each subintervals if Δt, a change in time, and the height of the approximating rectangle is a velocity. The product of time and velocity is distance.

This interprets the area above the graph $y = v(t)$ as the **negative** of the distance traveled between $t = 0$ and $t = 1$. The sign tells you that the object is traveling in the opposite direction on the axis.

Displacement versus Distance Traveled

Displacement versus Distance Traveled

If we add these two numbers together, we get the **displacement** of the object over the interval $[0, 5]$. That is, you get the distance between where the object started and where it ended. This adds distance to the right and subtracts distance to the left

Displacement versus Distance Traveled

If we subtract these two numbers, you get the the **distance traveled** by the object over the interval $[0, 5]$. That is, you get the distance the object traveled to the right plus the distance traveled to the left. Here, we just ignore the direction of travel and add the distances together.

The displacement and the distance traveled are generally not the same thing.