Newton's Method

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Outline

Newton’s Method

Newton's Method

Newton's Method is a numerical technique for finding roots of equations. This technique is a part of the mathematical discipline called numerical analysis.

The idea is to start at a point near a root and construct the tangent line at that point on the curve. You then find the root of the tangent line. You then repeat the process iteratively. When the numbers produced approach a limit, that's the root of the equation.

Procedure for Newton's Method

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Start with a function $f(x)$ for which we want to find a root.

Start with a point x_0 near a root. The point on the curve then is $(x_0, f(x_0))$ and the equation of the tangent line is

$$y = f(x_0) + f'(x_0)(x - x_0)$$

We set y equal to zero to find the root of this equation.

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Call this point x_1.

Procedure for Newton's Method

Repeating this iteratively, we get the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example

Example 1

Example

Approximate the positive root of the equation $f(x) = x^2 - 5 = 0$.

Example 1

Solution

We start with a guess near the root of f. Let's take $x_0 = 2$.

Now we use the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

to compute x_1, x_2, x_3, and so on.

Example 1

Solution

We compute:

x_0	2
x_1	2.25
x_2	2.23611
x_3	2.236067978
x_4	2.236067978

Once the numbers stop changing, this is the root.

Failures of Newton's Method

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

Failures of Newton's Method

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.

Failures of Newton's Method

2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root. This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.

Failures of Newton's Method

3. The approximations may fail to approach a root entirely. In the next example, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

Example

Example

Consider the function $f(x) = x^3 - 2x + 2$ Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Example

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 3$.

Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Example

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$. Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

See the sketch on the next slide.

Example

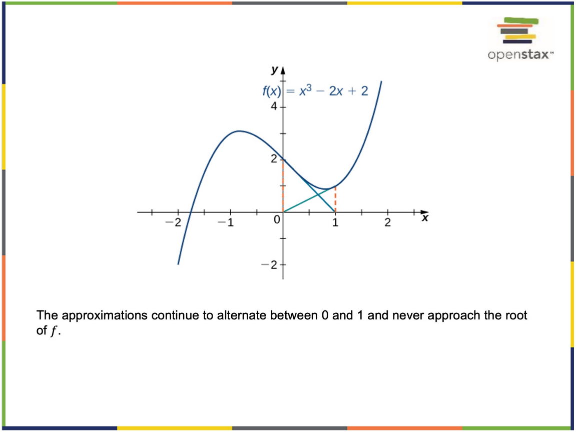

Figure: Failure of L'Hôpital's Rule