L'Hôpital's Rule

William M. Faucette

University of West Georgia

Outline

Indeterminate Forms

There are (at least) seven **indeterminate forms**. These are limits that look like

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

When you have a limit of one of these forms, you have to put forth more effort to compute the limit.

L'Hôpital's Rule: $0/0$

L'Hôpital's Rule: 0/0

This is the basic form of L'Hôpital's Rule to deal with the indeterminate form 0/0.

L'Hôpital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Examples

Example 1

Example

Compute

$$\lim_{x \to -2} \frac{x+2}{x^2-4}.$$

Example 1

Solution

We notice that $\lim_{x\to -2}(x+2) = \lim_{x\to -2}(x^2-4) = 0$, *so this limit looks like* $0/0$. *We apply L'Hôpital's Rule:*

$$\begin{aligned}\lim_{x\to -2}\frac{x+2}{x^2-4} &= \lim_{x\to -2}\frac{(x+2)'}{(x^2-4)'}\\ &= \lim_{x\to -2}\frac{1}{2x}\\ &= -\frac{1}{4}.\end{aligned}$$

Example 2

Example

Compute

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Example 2

Solution

We notice that $\lim_{x\to 0} \sin x = \lim_{x\to 0} x = 0$, *so this limit looks like* $0/0$.
We apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1.$$

L'Hôpital's Rule: ∞/∞

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The approach of L'Hôpital's Rule for $0/0$ also works to deal with the indeterminate form ∞/∞.

L'Hôpital's Rule

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Examples

Example 3

Example

Compute

$$\lim_{x \to \infty} \frac{5x^3 - 2x}{7x^3 + 3}.$$

Example 3

Solution

We notice that $\lim_{x\to\infty}(5x^3 - 2x) = \lim_{x\to -2}(7x^3 + 3) = \infty$, *so this limit looks like* ∞/∞. *We apply L'Hôpital's Rule:*

$$\lim_{x\to\infty} \frac{5x^3 - 2x}{7x^3 + 3} = \lim_{x\to\infty} \frac{(5x^3 - 2x)'}{(7x^3 + 3)'}$$
$$= \lim_{x\to\infty} \frac{15x^2 - 2}{21x^2}.$$

(You might ponder how we know that $\lim_{x\to\infty}(5x^3 - 2x) = \infty$.)

Example 3

Solution

We notice this limit still looks like ∞/∞, so we apply L'Hôpital's Rule again.

$$\lim_{x\to\infty}\frac{15x^2-2}{21x^2} = \lim_{x\to\infty}\frac{30x}{42x} = \lim_{x\to\infty}\frac{5}{7} = \frac{5}{7}.$$

Example 4

Example

Compute

$$\lim_{x \to \infty} \frac{\ln x}{x}.$$

Example 4

Solution

We notice this limit looks like ∞/∞, so we apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Example 5

Example

Compute

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}.$$

Example 5

Solution

This limit looks like ∞/∞, so we apply L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to\infty}\frac{(\ln x)^2}{x} &= \lim_{x\to\infty}\frac{[(\ln x)^2]'}{(x)'}\\ &= \lim_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{2\ln x}{x}\\ &= 2\lim_{x\to\infty}\frac{\ln x}{x} = 0,\end{aligned}$$

where we have used the result from the last example. Notice this problem requires using L'Hôpital's Rule twice.

L'Hôpital's Rule: $\infty - \infty$

L'Hôpital's Rule: $\infty - \infty$

In order to deal with indeterminate forms of the form $\infty - \infty$ you do some sort of algebraic manipulation to put this into the form $0/0$ or ∞/∞.

Example

Example 6

Example

Compute

$$\lim_{x\to 1^-}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right).$$

Example 6

Solution

We notice that $\lim_{x\to 1^-} \frac{1}{x-1} = \lim_{x\to 1^-} \frac{1}{\ln x} = -\infty$*, so this limit has this indeterminate form.*

$$\lim_{x\to 1^-} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \lim_{x\to 1^-} \frac{\ln(x) - (x-1)}{(x-1)\ln x}$$
$$= \lim_{x\to 1^-} \frac{\ln(x) - x + 1}{(x-1)\ln x}.$$

This limit has the indeterminate form $0/0$*.*

Example 6

Solution

Since our computation has led us to an indeterminate of the form $0/0$, *we apply L'Hôpital's Rule:*

$$\begin{aligned}\lim_{x\to 1^-}\frac{\ln(x)-x+1}{(x-1)\ln x} &= \lim_{x\to 1^-}\frac{(\ln(x)-x+1)'}{[(x-1)\ln x]'}\\ &= \lim_{x\to 1^-}\frac{\frac{1}{x}-1}{(x-1)'\ln x+(x-1)(\ln x)'}\\ &= \lim_{x\to 1^-}\frac{\frac{1}{x}-1}{(1)\ln x+(x-1)(\frac{1}{x})}\end{aligned}$$

Example 6

Solution

We continue by first multiplying by x/x to eliminate the complex fraction:

$$\lim_{x\to 1^-} \frac{\frac{1}{x}-1}{\ln x + (x-1)(\frac{1}{x})} \cdot \frac{x}{x} = \lim_{x\to 1^-} \frac{1-x}{x\ln x + x - 1}.$$

This looks like $0/0$, so we apply L'Hôpital's Rule again:

$$\begin{aligned} \lim_{x\to 1^-} \frac{1-x}{x\ln x + x - 1} &= \lim_{x\to 1^-} \frac{(1-x)'}{(x\ln x + x - 1)'} \\ &= \lim_{x\to 1^-} \frac{-1}{\ln(x)+2} = -\frac{1}{2}. \end{aligned}$$

L'Hôpital's Rule: $\infty \cdot 0$

L'Hôpital's Rule: $\infty \cdot 0$

In order to deal with indeterminate forms of the form $\infty \cdot 0$ you invert either factor to convert this form into $0/0$ or ∞/∞.

Example

Example 7

Example

Compute

$$\lim_{x \to 0^+} x \ln(x)$$

Example 7

Solution

We notice that $\lim_{x\to 0^+} x = 0$ *and* $\lim_{x\to 0^+} \ln(x) = -\infty$*, so this limit has the indeterminate form* $0 \cdot (-\infty)$*. (This is the same indeterminate form as* $\infty \cdot 0$*.)*

We invert one of these factors and divide

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x}.$$

This looks like $-\infty/\infty$*. So, we apply L'Hôpital's Rule.*

Example 7

Solution

Continuing from the preceding slide by applying L'Hôpital's Rule, we get

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x)}{1/x} &= \lim_{x\to 0^+} \frac{(\ln(x))'}{(1/x)'} \\
&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\
&= \lim_{x\to 0^+} -x \\
&= 0.
\end{aligned}$$

Exponential Indeterminate Forms

The remaining three indeterminate forms are all exponentials. These are all dealt with in the same way.

Taking the natural logarithm of the expression yields and expression of the form $\infty \cdot 0$, which you compute as we did before.

When you're finished, you have to remember that you took the natural logarithm to get this answer, so you must take the result and exponentiate it to get the original limit.

L'Hôpital's Rule: 0^0

L'Hôpital's Rule: 0^0

To deal with limits of the form 0^0, you take the natural logarithm of the limit. This converts the limit to one of the form $0 \cdot \infty$.

Compute this limit as before, then exponentiate to get the value of the original limit.

Example

Example 8

Example

Compute

$$\lim_{x \to 0^+} x^x$$

Example 8

Solution

We notice that this limit looks like 0^0*, which is an exponential indeterminate form. We set* $L = \lim_{x\to 0^+} x^x$*. Now, we take the natural logarithm of this equation.*

$$\begin{aligned} \ln(L) &= \ln(\lim_{x\to 0^+} x^x) \\ &= \lim_{x\to 0^+} \ln(x^x) \\ &= \lim_{x\to 0^+} x\ln(x). \end{aligned}$$

Example 8

Solution

This limit has the indeterminate form $0 \cdot (-)\infty$*, which we compute using the process presented before.*

$$\begin{aligned} \ln(L) &= \lim_{x \to 0^+} x \ln(x) \\ &= 0. \\ L &= e^0 = 1. \end{aligned}$$

So, the original limit is 1.

L'Hôpital's Rule: 1^{∞}

L'Hôpital's Rule: 1^∞

To deal with limits of the form 1^∞, you take the natural logarithm of the limit. This converts the limit to one of the form $\infty \cdot 0$.

Compute this limit as before, then exponentiate to get the value of the original limit.

Example

Example 9

Example

Compute

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Example 9

Solution

We note that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1$*, so this limit looks like* 1^∞*, which is an exponential indeterminate form. We set* $L=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$*. Now, we take the natural logarithm of this equation.*

$$\begin{aligned}\ln(L) &= \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)\\ &= \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x\\ &= \lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right).\end{aligned}$$

Example 9

Solution

This limit has the form $\infty \cdot 0$*, which we compute using the process presented before.*

So, we have

$$\begin{aligned}\ln(L) &= \lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

Now, this limit looks like $0/0$ *and we can apply L'Hôpital's Rule directly.*

Example 9

Solution

We apply L'Hôpital's Rule for the indeterminate form $0/0$*:*

$$\begin{aligned}\ln(L) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \\ &= \lim_{x\to\infty} \frac{\left[\ln\left(1+\frac{1}{x}\right)\right]'}{(1/x)'} \\ &= \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}(1/x)'}{(1/x)'} \\ &= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} = 1.\end{aligned}$$

So, $L = e^1 = e$*. The original limit is* e*.*

L'Hôpital's Rule: ∞^0

L'Hôpital's Rule: ∞^0

To deal with limits of the form ∞^0, you take the natural logarithm of the limit. This converts the limit to one of the form $0 \cdot \infty$.

Compute this limit as before, then exponentiate to get the value of the original limit.

Example

Example 10

Example

Compute
$$\lim_{x \to \infty} x^{1/x}.$$

Example 10

Solution

We note that $\lim_{x\to\infty}(1/x) = 0$*, so this limit looks like* ∞^0*, which is an exponential indeterminate form. We set* $L = \lim\limits_{x\to\infty} x^{1/x}$.

Now, we take the natural logarithm of this equation.

$$\begin{aligned}\ln(L) &= \ln\left(\lim_{x\to\infty} x^{1/x}\right)\\ &= \lim_{x\to\infty} \ln\left(x^{1/x}\right)\\ &= \lim_{x\to\infty} \frac{1}{x}\ln(x)\\ &= \lim_{x\to\infty} \frac{\ln(x)}{x}.\end{aligned}$$

Example 10

Solution

We have already computed this limit in a preceding example, where we found that $\lim_{x\to\infty}(\ln x/x) = 0$*.*

So, we have

$$L = e^0 = 1$$

as our original limit.