Applied Optimization Problems

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Outline

Solving Applied Optimization Problems

One common application of calculus is calculating the minimum or maximum value of a function. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Applied Optimization Problems

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The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied.

Solving Applied Optimization Problems

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1. Read the problem. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. Draw a picture. Label any part that may be important to the problem.
3. Introduce variables. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).

Solving Applied Optimization Problems

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5. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
6. Write any equations relating the independent variables in the formula from step 5. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
7. Identify the domain of consideration for the function in step 6 based on the physical problem to be solved.
8. Locate the maximum or minimum value of the function from step 6. This step typically involves looking for critical points and evaluating a function at endpoints.

Examples

Example 1

Example

Show that among all rectangles with an 8 m perimeter, the one with largest area is a square.

Example 1

Solution

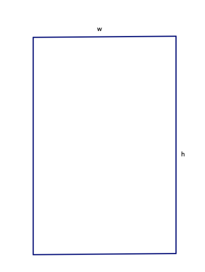
The first thing you do is read the problem and express it in your own words. You want to take a rectangle of perimeter 8 meters and find the one with the largest area. We want to show that figure is a square.

Example 1

Solution

Next, you draw a picture. Below is a sketch of a rectangle. Then we label the variables. I've labeled the width w and the height h.

Figure: *Sketch of Rectangle*

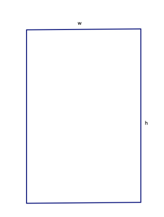


Example 1

Solution

Next, you write down what you're trying to maximize in terms of the variables. We are trying to maximize the area, so we write $A = wh$.

Figure: *Sketch of Rectangle*



Example 1

Solution

The equation $A = wh$ has too many variables, so we need a relationship between w and h to eliminate one of those variables.

The relationship is that the perimeter of the rectangle is 8 meters. This gives us $2w + 2h = 8$, or $w + h = 4$. Solving this for h, we have $h = 4 - w$.

Substitute this expression into the area equation. That gives us the area of the rectangle as a function of the width of the rectangle.

$$A = wh = w(4 - w)$$

Example 1

Solution

It's always nice to know the domain of the function. In this function, both h and w are lengths, so that must be at least zero. So, $w \geq 0$ and $h = 4 - w \geq 0$. This tells us the $0 \leq w \leq 4$. Now we have a continuous function on a closed interval.

So, we now have just an ordinary optimization problem.

Example 1

Solution

We want to find the maximum of the function $A = w(4 - w)$ *on the interval* $[0, 4]$.

We take the derivative and find the critical points

$$\begin{aligned} A &= w(4 - w) = -w^2 + 4w \\ \frac{dA}{dw} &= -2w + 4 \\ \frac{dA}{dw} &= 0 \text{ if } w = 2. \end{aligned}$$

So, we have three points to look at. We have this critical point and the two endpoints: $w = 0, 2, 4$.

Example 1

Solution

We form a T-chart and evaluate the function $A = w(4 - w)$ at each of these three points.

w	$A(w)$
0	0
2	4
4	0

So, the maximum area occurs when $w = 2$. When $w = 2$, $h = 4 - w = 4 - 2 = 2$. So, both the width and height are 2 meters of the rectangle of largest area. The rectangle of maximum area is a square.

Example 2

Example

A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

Example 2

Solution

The first thing you do is read the problem and express it in your own words. You have a rectangle with base on the x-axis and the top two corners on the graph $y = 12 - x^2$.

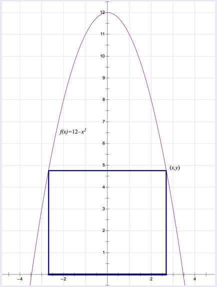
You want to find the area and dimensions of the largest rectangle. So, you're trying to maximize the area of the rectangle

Example 2

Solution

Next, you draw a picture. Below is a sketch of a rectangle.

Figure: *Sketch of Rectangle Inscribed in Parabola*



Example 2

Solution

Then we label the variables. I've labeled the upper right corner of the rectangle as the point (x, y).

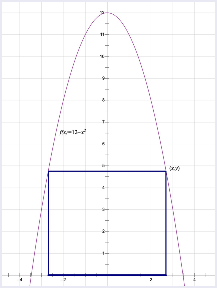


Figure: *Sketch of Rectangle Inscribed in Parabola*

Example 2

Solution

Next, you write down what you're trying to maximize in terms of the variables. The height of the rectangle is y and the width of the rectangle is $2x$, so the area is the product of these, $A = 2xy$.

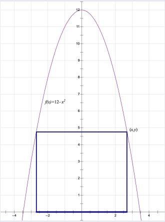


Figure: *Sketch of Rectangle Inscribed in Parabola*

Example 2

Solution

The equation $A = 2xy$ *has too many variables, so we need a relationship between* x *and* y *to eliminate one of those variables.*

The relationship is that the point lies on the graph. So, $y = 12 - x^2$.

Substitute this expression into the area equation. That gives us the area of the rectangle as a function of the width of x.

$$A = 2xy = 2x(12 - x^2).$$

Example 2

Solution

It's always nice to know the domain of the function. In this function, both x and y are lengths, so that must be at least zero. So, $x \geq 0$ and $y = 12 - x^2 \geq 0$.

This tells us the $0 \leq x \leq \sqrt{12} = 2\sqrt{3}$. Now we have a continuous function on a closed interval.

So, we now have just an ordinary optimization problem.

Example 2

Solution

We want to find the maximum of the function $A = 2x(12 - x^2)$ *on the interval* $[0, 2\sqrt{3}]$.

We take the derivative and find the critical points

$$\begin{aligned} A &= 2x(12 - x^2) = 24x - 2x^3 \\ \frac{dA}{dx} &= 24 - 6x^2 \\ \frac{dA}{dx} &= 0 \text{ if } x = 2. \end{aligned}$$

So, we have three points to look at. We have this critical point and the two endpoints: $x = 0, 2, 2\sqrt{3}$.

Example 2

Solution

We form a T-chart and evaluate the function $A = 2x(12 - x^2)$ *at each of these three points.*

x	$A(x)$
0	0
2	32
$2\sqrt{3}$	0

So, the maximum area occurs when $x = 2$*. When* $x = 2$*,* $y = 12 - x^2 = 12 - 2^2 = 8$*. So the largest rectangle has width* $2x = 4$ *and height* 8*. The largest area is 32.*

Example 3

Example

Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (Hint: $A = \frac{1}{2}ab\sin\theta$.)

Example 3

Solution

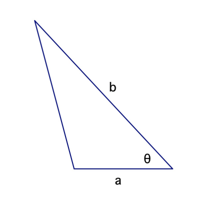
The first thing you do is read the problem and express it in your own words. Here we have a triangle with side lengths a and b and included angle of measure θ *(radians). We want to find the area of the largest triangle. We're also given a relationship between the area, A, and the quantities a, b, and* θ*.*

Example 3

Solution

Next, you draw a picture. Below is a sketch of a triangle. Then we label the side lengths a, b, and the angle θ, which are given to us in the problem.

Figure: *Sketch of Triangle*

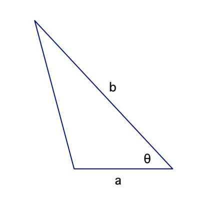


Example 3

Solution

Next, you write down what you're trying to maximize in terms of the variables. We are trying to maximize the area, and fortunately the problem gives us the relationship: $A = \frac{1}{2}ab\sin\theta$*.*

Figure: *Sketch of Triangle*



Example 3

Solution

The equation $A = \frac{1}{2}ab\sin\theta$ *looks like it has too many variables, but it doesn't. The quantities a and b are constants. The only variable here is* θ.

So, we have the area of the triangle as a function of θ.

$$A = \frac{1}{2}ab\sin\theta$$

Example 3

Solution

It's always nice to know the domain of the function. Since θ *is an angle in a triangle and all the angles in a triangle add up to* $180^\circ = \pi$ *radians. So,* $0 \le \theta \le \pi$.

So, we now have just an ordinary optimization problem.

Example 3

Solution

We want to find the maximum of the function $A = \frac{1}{2}ab\sin\theta$ *on the interval* $[0, \pi]$.

We take the derivative and find the critical points

$$\begin{aligned} A &= \frac{1}{2}ab\sin\theta \\ \frac{dA}{d\theta} &= \frac{1}{2}ab\cos\theta \\ \frac{dA}{d\theta} &= 0 \textit{ if } \theta = \frac{\pi}{2}. \end{aligned}$$

So, we have three points to look at. We have this critical point and the two endpoints: $\theta = 0, \frac{\pi}{2}, \pi$.

Example 3

Solution

We form a T-chart and evaluate the function $A = \frac{1}{2}ab\sin\theta$ *at each of these three points.*

θ	$A(\theta)$
0	0
$\frac{\pi}{2}$	$\frac{1}{2}ab$
π	0

So, the maximum area occurs when $\theta = \pi/2$*—that is, the largest triangle is a right triangle. The area of the largest triangle is* $\frac{1}{2}ab$*—one-half base times height.*

Example 4

Example

You are designing a rectangular poster to contain 50 in^2 of printing with a 4 inch margin at the top and bottom and a 2 inch margin at each side. What overall dimensions will minimize the amount of paper used?

Example 4

Solution

The first thing you do is read the problem and express it in your own words. You're making a poster with 50 in^2 *of printed area and margins at the top of bottom of 4 inches and margins on the left and right of 2 inches. You want to minimize the paper used—the area of the whole poster.*

Example 4

Solution

Next, you draw a picture. Below is a sketch of the poster. Then we label the variables. I've let x *be the width of the printed area of the poster and let* y *be the height of the printed area of the poster.*

See the sketch on the next slide.

Example 4

Solution

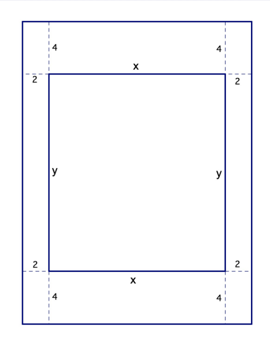


Figure: *Sketch of Poster*

Example 4

Solution

Next, you write down what you're trying to minimize in terms of the variables. We are trying to minimize the area of the whole poster. The width of the poster board is x *plus the two side margins, so that's* $x + 4$*. The height of the poster board is* y *plus the top and bottom margins, so that's* $y + 8$*.*

So, we are trying to minimize the function

$$A = (x + 4)(y + 8).$$

Example 4

Solution

The equation $A = (x+4)(y+8)$ *has too many variables, so we need a relationship between the variables. The relationship is given by the fact that the printed area of the poster is 50* in^2*. So, we have* $xy = 50$*. Solving this for* y *and substituting this into our function, we get*

$$\begin{aligned} A &= (x+4)(y+8) \\ &= (x+4)\left(\frac{50}{x}+8\right) \\ &= 82 + 8x + \frac{200}{x}. \end{aligned}$$

Example 4

Solution

It's always nice to know the domain of the function. The variables x is a length, so we must have $x \geq 0$. Since $x = 0$ is not in the domain of the function, we must actually have $x > 0$.

So, we now have just an ordinary optimization problem, but it's not on a closed finite interval.

Example 4

Solution

We want to find the maximum of the function

$$A = 82 + 8x + \frac{200}{x}$$

on the interval $(0, \infty)$*.*

We take the derivative and find the critical points

$$A = 82 + 8x + \frac{200}{x}$$
$$\frac{dA}{dx} = 8 - \frac{200}{x^2}.$$

Example 4

Solution

From the last slide, we found

$$\frac{dA}{dx} = 8 - \frac{200}{x^2}.$$

This derivative is zero at $x = 5$. *(The derivative is undefined at* $x = 0$, *but so is the function!)*

So, we have only have one critical point to look at: $x = 5$.

Example 4

Solution

Using a T-chart here won't work because we don't have a continuous function on a closed finite interval. We have to do something else.

If we look at the sign of the first derivative, we find that $dA/dx < 0$ *on the interval* $(0, 5)$ *and* $dA/dx > 0$ *on the interval* $(5, \infty)$*.*

This shows that A *has an absolute minimum at* $x = 5$ *by the First Derivative Test.*

Example 4

Solution

When $x = 5$ *inches,* $y = 50/5 = 10$ *inches.*

The smallest poster has width $x + 4 = 5 + 4 = 9$ *inches and height* $y + 8 = 10 + 8 = 18$ *inches.*

Example 5

Example

A piece of cardboard measures 10 in by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

Find the dimensions and the volume of the box of largest volume.

Example 5

Solution

The first thing you do is read the problem and express it in your own words.

You have this cardboard template, originally 10 inches by 15 inches. You're going to fold up the four sides along the dotted lines, then fold over the lid, to form a closed box.

You want to find the dimensions and volume of the largest box.

Example 5

Solution

Next, you draw a picture.

Fortunately the picture is given to us in the problem.

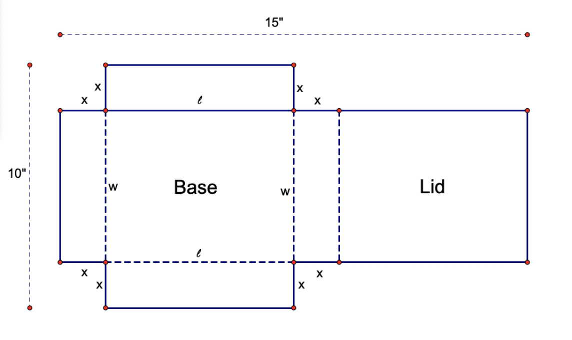
The variable x is also labeled in the problem. I'm also going to label the length ℓ and width w of the base of the box.

See the sketch on the next slide.

Example 5

Solution

Figure: *Sketch of Cardboard Template*



Example 5

Solution

Next, you write down what you're trying to maximize in terms of the variables. We are trying to maximize the volume of the box. This is given by $V = \ell w x$.

Example 5

Solution

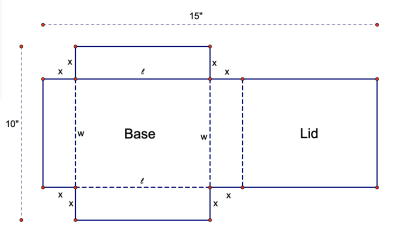
The equation $V = \ell wx$ has too many variables, so we need equations expressing ℓ and w in terms of x.

Example 5

Solution

Looking at the sketch and the variables, if you look horizontally, you have $x + \ell + x + \ell = 15$. *So,* $\ell = \frac{15}{2} - x$.

Figure: *Sketch of Cardboard Template*

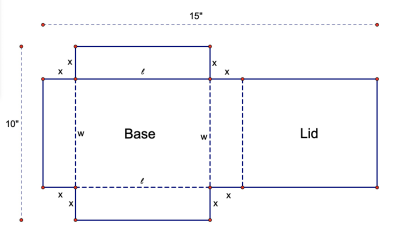


Example 5

Solution

Looking at the sketch and the variables, if you look vertically, you have $x + w + x = 10$*. So,* $w = 10 - 2x$.

Figure: *Sketch of Cardboard Template*



So, our function is given by $V = (\frac{15}{2} - x)(10 - 2x)x$.

Example 5

Solution

It's always nice to know the domain of the function. Since x, ℓ, *and* w *are all lengths, they must be nonnegative. So,* $x \geq 0$, $\ell = \frac{15}{2} - x \geq 0$, *and* $w = 10 - 2x \geq 0$.

This tells us that $x \geq 0$, $x \leq \frac{15}{2}$, *and* $x \leq 5$.

Putting these together, we get $0 \leq x \leq 5$.

So, we now have just an ordinary optimization problem.

Example 5

Solution

We want to find the maximum of the function $V = (\frac{15}{2} - x)(10 - 2x)x$ *on the interval* $[0, 5]$.

We take the derivative and find the critical points

$$\begin{aligned} V &= \left(\frac{15}{2} - x\right)(10 - 2x)x \\ &= 2x^3 - 25x^2 + 75x \\ \frac{dV}{dx} &= 6x^2 - 50x + 75. \end{aligned}$$

Example 5

Solution

From the last slide, we have

$$\frac{dV}{dx} = 6x^2 - 50x + 75.$$

We set this equal to zero and use the quadratic formula to find that $dV/dx = 0$ *when* $x = \frac{25}{6} \pm \frac{5\sqrt{7}}{6}$ *Only one of these is in our domain:* $\frac{25}{6} - \frac{5\sqrt{7}}{6}$.

So, we have three points to look at. We have this critical point and the two endpoints: $x = 0, \frac{25}{6} - \frac{5\sqrt{7}}{6}, 5$.

Example 5

Solution

We form a T-chart and evaluate the function $V = (\frac{15}{2} - x)(10 - 2x)x$ *at each of these three points.*

x	$V(x)$
0	0
$\frac{25}{6} - \frac{5\sqrt{7}}{6}$	$\frac{625}{27} + \frac{875\sqrt{7}}{54}$
5	0

So, the maximum volume occurs when $x = \frac{25}{6} - \frac{5\sqrt{7}}{6} \approx 1.96$ *in. The box has length approximately 5.54 in, width approximately 6.08 in and height approximately 1.96 in. The volume of this largest box is approximately* 66.02 in^3.