Limits Involving Infinity; Asymptotes of Graphs

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Outline

Finite Limits as $x \to \pm\infty$

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Definition

We say a function f has a **limit at infinity**, if there exists a real number L such that for all $\epsilon > 0$, there exists $N > 0$ such that

$$|f(x) - L| < \epsilon$$

for all $x > N$. In that case, we write

$$\lim_{x\to\infty} f(x) = L.$$

Finite Limits as $x \to \pm\infty$

Definition

We say a function f has a **limit at negative infinity**, if there exists a real number L such that for all $\epsilon > 0$, there exists $N < 0$ such that

$$|f(x) - L| < \epsilon$$

for all $x < N$. In that case, we write

$$\lim_{x \to -\infty} f(x) = L.$$

Examples

Example 1

Example

The function $f(x) = \frac{1}{x}$ has the limit zero as x goes to infinity or to minus infinity.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Example 2

Example

$$\lim_{x\to\infty}\left(8-\frac{1}{x^2}\right)=\lim_{x\to\infty}8-\lim_{x\to\infty}\frac{1}{x^2}=\lim_{x\to\infty}8-\lim_{x\to\infty}\left(\frac{1}{x}\right)^2$$
$$=\lim_{x\to\infty}8-\left(\lim_{x\to\infty}\frac{1}{x}\right)^2=8-(0)^2=8.$$

Example 3

Example

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{2+\frac{1}{x}} &= \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty}\left(2+\frac{1}{x}\right)} \\
&= \frac{\lim_{x\to\infty} 1}{(\lim_{x\to\infty} 2) + \lim_{x\to\infty}\left(\frac{1}{x}\right)} \\
&= \frac{1}{2+0} = \frac{1}{2}.
\end{aligned}$$

Example 4

Example

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \left(\frac{2x+3}{5x+7}\right) \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \left(\frac{2+\frac{3}{x}}{5+\frac{7}{x}}\right)$$

$$= \frac{\lim_{x\to\infty} \left(2+\frac{3}{x}\right)}{\lim_{x\to\infty} \left(5+\frac{7}{x}\right)}$$

$$= \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{3}{x}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}.$$

Example 5

Example

$$
\begin{aligned}
\lim_{x\to\infty}\frac{9x^3+x}{2x^4+5x^2-x+6} &= \lim_{x\to\infty}\frac{9x^3+x}{2x^4+5x^2-x+6}\cdot\frac{1/x^4}{1/x^4}\\
&= \lim_{x\to\infty}\frac{\frac{9}{x}+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}}\\
&= \frac{\lim_{x\to\infty}\frac{9}{x}+\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{5}{x^2}-\lim_{x\to\infty}\frac{1}{x^3}+\lim_{x\to\infty}\frac{6}{x^4}}\\
&= \frac{0+0}{2+0-0+0}=0.
\end{aligned}
$$

Example 6

Example

Find **(a)** $\lim_{x\to\infty} \sin(1/x)$ and **(b)** $\lim_{x\to\infty} x\sin(1/x)$

Example 6

Solution

a *Let* $t = 1/x$. *Then*

$$\lim_{x \to \infty} \sin(1/x) = \lim_{t \to 0^+} \sin(t) = 0.$$

b *Let* $t = 1/x$. *Then*

$$\lim_{x \to -\infty} x \sin(1/x) = \lim_{t \to 0^-} \frac{\sin(t)}{t} = 1.$$

Example 7

Example

Find $\lim_{x\to\infty}\left(x-\sqrt{x^2+4}\right)$.

Example 7

Solution

We compute

$$\begin{aligned}\lim_{x\to\infty}\left(x-\sqrt{x^2+4}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+4}\right)\cdot\frac{x+\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\\ &= \lim_{x\to\infty}\frac{x^2-(x^2+4)}{x+\sqrt{x^2+4}}\\ &= \lim_{x\to\infty}\frac{-4}{x+\sqrt{x^2+4}}\\ &= 0.\end{aligned}$$

Horizontal Asymptotes

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an **asymptote** of the graph.

Horizontal Asymptotes

Definition

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Example 8

Example

Find the horizontal asymptotes, if any, of the function $f(x) = \arctan x$.

Example 8

Solution

We need to compute $\lim_{x\to\infty} \arctan(x)$ *and* $\lim_{x\to-\infty} \arctan(x)$*. In order to do this, we will use the graph* $y = \arctan(x)$*.*

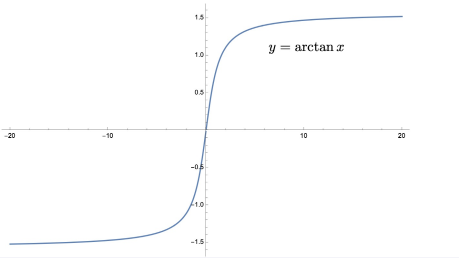

Figure: *Graph of* $y = \arctan x$

Example 8

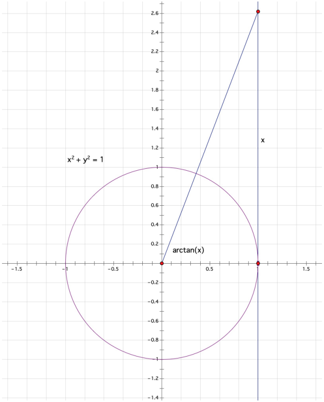

Figure: arctan(x) for x positive

To compute $\lim_{x\to\infty} \arctan(x)$, we draw the unit circle with a line tangent to the circle at the point $(1, 0)$. If x is the distance from the x-axis upward along the tangent line, the labeled angle is then $\arctan(x)$.

From the sketch in the figure, we see that if x be comes very, very large, the angle $\arctan(x)$ goes to $\pi/2$.

Example 8

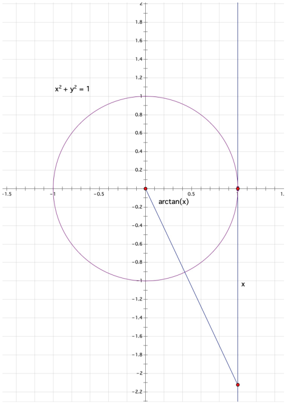

Figure: arctan(x) for x negative

To compute $\lim_{x \to -\infty} \arctan(x)$, we draw the unit circle with a line tangent to the circle at the point $(1, 0)$. If x is minus the distance from the x-axis downward along the tangent line, the labeled angle is then $\arctan(x)$.

From the sketch in the figure, we see that if x be comes very, very large, the angle $\arctan(x)$ goes to $-\pi/2$.

Example 9

Example

Find the horizontal asymptotes, if any, of the function $f(x) = \frac{x}{e^x}$.

Example 9

Solution

We compute the limit of f as x approaches infinity using the graph of the function. From the graph, we see that x/e^x goes to zero as x goes to infinity.

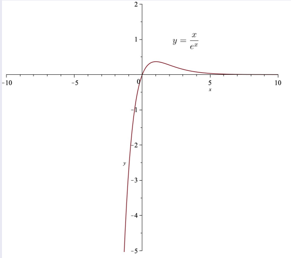

Figure: *Graph of $y = x/e^x$*

Example 9

Solution

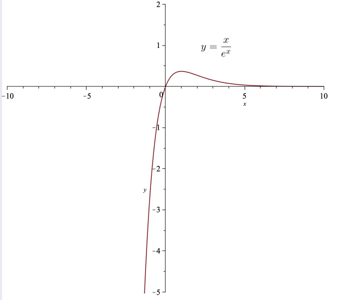

Figure: *Graph of $y = x/e^x$*

So the line y=0—the x-axis—in an asymptote of the graph.

Infinite Limits at Infinity

Infinite Limits at Infinity

Definition

We say a function f has a **infinite limit at infinity** and write

$$\lim_{x \to \infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$.

Infinite Limits at Infinity

Definition

We say a function f has a **negative infinite limit at infinity** and write

$$\lim_{x \to \infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

$$f(x) < M$$

for all $x > N$.

Similarly we can define limits as $x \to -\infty$.

End Behavior

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

End Behavior Of Polynomial Functions

End Behavior Of Polynomial Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. Then take the limit as $x \to \pm\infty$. The result then depends on the degrees of the polynomials involved.

End Behavior Of Polynomial Functions

Suppose we have a rational function $f(x) = p(x)/q(x)$, where p and q are polynomials. There are three possibilities:

1. If $\deg p < \deg q$, then there is a horizontal asymptote at $y = 0$.

This is because

$$\lim_{x \to \pm\infty} f(x) = 0.$$

End Behavior Of Polynomial Functions

Suppose we have a rational function $f(x) = p(x)/q(x)$, where p and q are polynomials. There are three possibilities:

2. If $\deg p = \deg q$, then there is a horizontal asymptote at $y = p_n/q_n$, where p_n is the leading coefficient of p and q_n is the leading coefficient of q.

This is because

$$\lim_{x \to \pm\infty} f(x) = \frac{p_n}{q_n}.$$

End Behavior Of Polynomial Functions

Suppose we have a rational function $f(x) = p(x)/q(x)$, where p and q are polynomials. There are three possibilities:

3. If $\deg p > \deg q$, then there is no horizontal asymptote.

This is because

$$\lim_{x \to \pm\infty} f(x) = \pm\infty.$$

Oblique Asymptotes

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique asymptote**, a slanted line which the graph approaches as x goes to infinity.

We find an equation for the asymptote by dividing the numerator by the denominator to express f as a linear function plus a remainder that goes to zero as x goes to $\pm\infty$.

Example

Example 15

Example

Find the oblique asymptote for the function

$$f(x) = \frac{x^3 + 1}{x^2}.$$

Example 15

Solution

We do long division of polynomials:

$$\begin{array}{r|l} & \quad\quad x \\ \hline x^2) & x^3 + 1 \\ & -x^3 \\ \hline & \quad\quad 1 \end{array}$$

From this, we see that

$$\frac{x^3+1}{x^2} = x + \frac{1}{x^2}.$$

Example 15

Solution

Since

$$\frac{x^3+1}{x^2} = x + \frac{1}{x^2}.$$

We have

$$\lim_{x\to\pm\infty} \frac{x^3+1}{x^2} - x = \lim_{x\to\pm\infty} \frac{1}{x^2} = 0.$$

We see that the function $f(x) = \dfrac{x^3+1}{x^2}$ *gets closer and closer to the function* $g(x) = x$ *for* x *very, very large.*
So, the oblique asymptote is $y = x$.

Procedure for Graphing $y = f(x)$

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Identify any asymptotes that may exist.
3. Find the derivatives y' and y''.
4. Find the critical points of f, if any, and identify the function's behavior at each one.
5. Find where the curve is increasing and where it is decreasing.
6. Find the inflection points, if any occur, and determine the concavity of the curve.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example

Example 4

Example

Graph the rational function

$$y = \frac{2x^2 + x - 1}{x^2 - 1}$$

using all the steps in the graphing procedure on the preceding slide.

Example 16

Solution

We first do the precalculus.
Our function is

$$y = \frac{2x^2 + x - 1}{x^2 - 1}.$$

Setting the denominator

$$x^2 - 1$$

equal to zero and solving, we find the domain of this function is all real numbers except $x = \pm 1$.

Example 16

Solution

We simplify the function to get

$$\frac{2x^2 + x - 1}{x^2 - 1} = \frac{(2x - 1)(x + 1)}{(x - 1)(x + 1)} = \frac{2x - 1}{x - 1}.$$

Setting $x = 0$*, we find a* y*-intercept at* $y = 1$*.*
Setting the numerator equal to zero, we find an x*-intercept at* $x = \frac{1}{2}$*.*

Example 16

Solution

Recall our function is

$$\frac{2x^2+x-1}{x^2-1} = \frac{(2x-1)(x+1)}{(x-1)(x+1)} = \frac{2x-1}{x-1}.$$

Taking the limit as $x \to \infty$, *we find horizontal asymptote at* $y = 2$.

There is a vertical asymptote at $x = 1$.

Since we canceled the factor $x + 1$, *there is no asymptote at* $x = -1$. *Instead, there is a hole in the graph at* $x = -1$. *(The function has a removable discontinuity at* $x = -1$.*)*

Example 16

Solution

Taking the derivative, we get

$$y' = \frac{(2x^2+x-1)'(x^2-1)-(2x^2+x-1)(x^2-1)'}{(x^2-1)^2}$$
$$= \frac{(4x+1)(x^2-1)-(2x^2+x-1)(2x)}{(x^2-1)^2}$$
$$= \frac{(4x^3+x^2-4x-1)-(4x^3+2x^2-2x)}{(x^2-1)^2}$$
$$= \frac{-x^2-2x-1}{[(x-1)(x+1)]^2} = \frac{-(x+1)^2}{(x-1)^2(x+1)^2} = -\frac{1}{(x-1)^2}.$$

Example 164

Solution

Taking the second derivative, we get $y'' = \frac{2}{(x-1)^3}$.

This derivative is never zero and is undefined at $x = 1$.

This derivative is negative for $x < 1$ *and positive for* $x > 1$.

So, $f'' < 0$ *on the interval* $(-\infty, 1)$ *and* $f'' > 0$ *on the interval* $(1, \infty)$.

So, f *is concave down on the interval* $(-\infty, 1)$ *and concave up on the interval* $(1, \infty)$.

Solution

Sketching the graph from the information we've gotten, we get

Figure: *Sketch of $y = f(x)$*