Indeterminate Forms and L'Hôpital's Rule
Basic Forms

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Outline

Indeterminate Forms and L'Hôpital's Rule

- As usual, you should read section 4.5 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Indeterminate Forms

There are (at least) seven **indeterminate forms**. These are limits that look like

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

When you have a limit of one of these forms, you have to put forth more effort to compute the limit.

This slide show will cover the first four indeterminate forms—the basic indeterminate forms.

L'Hôpital's Rule: $0/0$

This is the basic form of L'Hôpital's Rule to deal with the indeterminate form $0/0$.

L'Hôpital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 1

Example

Compute

$$\lim_{x\to -2} \frac{x+2}{x^2-4}.$$

Solution

We notice that $\lim_{x\to -2}(x+2) = \lim_{x\to -2}(x^2-4) = 0$, *so this limit looks like* $0/0$. *We apply L'Hôpital's Rule:*

$$\begin{aligned} \lim_{x\to -2} \frac{x+2}{x^2-4} &= \lim_{x\to -2} \frac{(x+2)'}{(x^2-4)'} \\ &= \lim_{x\to -2} \frac{1}{2x} \\ &= -\frac{1}{4}. \end{aligned}$$

Example 2

Example

Compute
$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Solution

We notice that $\lim_{x\to 0} \sin x = \lim_{x\to 0} x = 0$*, so this limit looks like* $0/0$*. We apply L'Hôpital's Rule:*

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{(\sin x)'}{x'} \\ &= \lim_{x\to 0} \frac{\cos x}{1} \\ &= \frac{\cos 0}{1} = 1.\end{aligned}$$

L'Hôpital's Rule: ∞/∞

The basic form of L'Hôpital's Rule for $0/0$ also works to deal with the indeterminate form ∞/∞.

L'Hôpital's Rule

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 3

Example

Compute

$$\lim_{x\to\infty} \frac{5x^3 - 2x}{7x^3 + 3}.$$

Solution

We notice that $\lim_{x\to\infty}(5x^3 - 2x) = \lim_{x\to -2}(7x^3 + 3) = \infty$, *so this limit looks like* ∞/∞. *We apply L'Hôpital's Rule:*

$$\lim_{x\to\infty} \frac{5x^3 - 2x}{7x^3 + 3} = \lim_{x\to\infty} \frac{(5x^3 - 2x)'}{(7x^3 + 3)'}$$
$$= \lim_{x\to\infty} \frac{15x^2 - 2}{21x^2}.$$

(You might ponder how we know that $\lim_{x\to\infty}(5x^3 - 2x) = \infty$.)

Example 3

Solution

We notice this limit still looks like ∞/∞, so we apply L'Hôpital's Rule again.

$$\lim_{x\to\infty} \frac{15x^2-2}{21x^2} = \lim_{x\to\infty} \frac{30x}{42x} = \lim_{x\to\infty} \frac{5}{7} = \frac{5}{7}.$$

Example 4

Example

Compute

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Solution

We notice this limit looks like ∞/∞, so we apply L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x}{x} &= \lim_{x\to\infty} \frac{1/x}{1} \\ &= \lim_{x\to\infty} \frac{1}{x} \\ &= 0.\end{aligned}$$

Example 5

Example

Compute

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}.$$

Solution

This limit looks like ∞/∞, so we apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln x)^2}{x} &= \lim_{x\to\infty} \frac{[(\ln x)^2]'}{(x)'} \\
&= \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x} \\
&= 2\lim_{x\to\infty} \frac{\ln x}{x} = 0,
\end{aligned}$$

where we have used the result from the last example. Notice this problem requires using L'Hôpital's Rule twice.

L'Hôpital's Rule: $\infty - \infty$

In order to deal with indeterminate forms of the form $\infty - \infty$ you do some sort of algebraic manipulation to put this into the form $0/0$ or ∞/∞.

Example 6

Example

Compute

$$\lim_{x\to 1^-}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right).$$

Solution

We notice that $\lim_{x\to 1^-}\frac{1}{x-1}=\lim_{x\to 1^-}\frac{1}{\ln x}=-\infty$, *so this limit has this indeterminate form.*

$$\lim_{x\to 1^-}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^-}\frac{\ln(x)-(x-1)}{(x-1)\ln x}=\lim_{x\to 1^-}\frac{\ln(x)-x+1}{(x-1)\ln x}.$$

This limit has the indeterminate form $0/0$.

Example 6

Solution

Since our computation has led us to an indeterminate of the form $0/0$*, we apply L'Hôpital's Rule:*

$$\begin{aligned}
\lim_{x\to 1^-} \frac{\ln(x) - x + 1}{(x-1)\ln x} &= \lim_{x\to 1^-} \frac{(\ln(x) - x + 1)'}{[(x-1)\ln x]'} \\
&= \lim_{x\to 1^-} \frac{\frac{1}{x} - 1}{(x-1)'\ln x + (x-1)(\ln x)'} \\
&= \lim_{x\to 1^-} \frac{\frac{1}{x} - 1}{(1)\ln x + (x-1)(\frac{1}{x})}
\end{aligned}$$

Example 6

Solution

We continue by first multiplying by x/x to eliminate the complex fraction:

$$\lim_{x\to 1^-} \frac{\frac{1}{x}-1}{\ln x + (x-1)(\frac{1}{x})} \cdot \frac{x}{x} = \lim_{x\to 1^-} \frac{1-x}{x\ln x + x - 1}.$$

This looks like $0/0$, so we apply L'Hôpital's Rule again:

$$\lim_{x\to 1^-} \frac{1-x}{x\ln x + x - 1} = \lim_{x\to 1^-} \frac{(1-x)'}{(x\ln x + x - 1)'}$$
$$= \lim_{x\to 1^-} \frac{-1}{\ln(x)+2} = -\frac{1}{2}.$$

L'Hôpital's Rule: $\infty \cdot 0$

In order to deal with indeterminate forms of the form $\infty \cdot 0$ you invert either factor to convert this form into $0/0$ or ∞/∞.

Example 7

Example

Compute

$$\lim_{x\to 0^+} x\ln(x)$$

Solution

We notice that $\lim_{x\to 0^+} x = 0$ *and* $\lim_{x\to 0^+} \ln(x) = -\infty$*, so this limit has the indeterminate form* $0\cdot(-\infty)$*. (This is the same indeterminate form as* $\infty\cdot 0$*.)*

We invert one of these factors and divide

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x}.$$

This looks like $-\infty/\infty$*. So, we apply L'Hôpital's Rule.*

Example 7

Solution

Continuing from the preceding slide by applying L'Hôpital's Rule, we get

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(x)}{1/x} &= \lim_{x\to 0^+}\frac{(\ln(x))'}{(1/x)'}\\ &= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+} -x\\ &= 0.\end{aligned}$$