The Mean Value Theorem

William M. Faucette

University of West Georgia

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Rolle's Theorem

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Rolle's Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Rolle's Theorem

What this says is that if a graph starts at a certain height and ends at that same height and is "nice" in between, then the derivative must be zero somewhere in between.

That is, if the function goes up, then it must come back down, so it must turn around, i.e. the derivative is zero. If the function goes down, then it must come back up, so it must turn around, i.e. the derivative is zero.

Example

Example 1

Example

Consider the function

$$f(x) = x^3 - 33x^2 + 216x = x(x-9)(x-24).$$

Show that this function is zero at $x = 0$, $x = 9$, and $x = 24$. Find the values of c guaranteed by Rolle's Theorem on the intervals $[0, 9]$ and $[9, 24]$.

Example 1

Solution

It's easy to see from the factored polynomial that it is zero at $x = 0$, $x = 9$, *and* $x = 24$. *So,* $f(0) = f(9) = 0$ *and* $f(9) = f(24) = 0$.

We take the derivative

$$y' = 3x^2 - 66x + 216 = 3(x - 4)(x - 18).$$

So, we see the derivative is zero at $x = 4$ *and* $x = 18$.

So, on the interval $[0, 9]$, *we have* $c = 4$ *and on the interval* $[9, 24]$ *we have* $c = 18$.

The Mean Value Theorem

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The Mean Value Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there is at least one number c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The Mean Value Theorem

The equation in the Mean Value Theorem is

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The fraction on the left is the average rate of change of f on the interval $[a, b]$. The value on the right is the instantaneous rate of change of f at $x = c$. So, what this says is that if a function has an average rate of change over and interval $[a, b]$, then at some point

If you average sixty miles per hour on a trip, at some point during the trip you must be going sixty miles per hour.

Example

Example 2

Example

Consider the function

$$f(x) = x^3 - x^2$$

on the interval $[-1, 2]$. Find the values of c guaranteed by the Mean Value Theorem.

Solution

This function is a polynomial, so it's continuous and differentiable everywhere. So, the function satisfies the hypotheses of the Mean Value Theorem.

We take the derivative: $f'(x) = 3x^2 - 2x$. *Then we compute the average rate of change of* f *on the interval:*

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - (-2)}{3} = \frac{6}{3} = 2.$$

So, we have to solve $f'(c) = 2$.

Solution

We set $f'(c) = 2$ *and solve for* c.

$$3c^2 - 2c = 2$$
$$3c^2 - 2c - 2 = 0$$
$$c = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)} = \frac{1}{3} \pm \frac{\sqrt{7}}{3}.$$

Both these numbers lie in the interval $[-1, 2]$*, so we have* $c = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$.

Important Corollaries

Corollary 1

The importance of the Mean Value Theorem is in the use of the fact that c exists in proving other results, not in actually computing c.

For example, we already know that if a function is constant, then its derivative is zero. One of the important results of the Mean Value Theorem is the converse of this statement.

Corollary 1

Corollary

If $f'(x) = 0$ *at each point* x *of an open interval* (a, b)*, then* $f(x) = C$ *for all* $x \in (a, b)$*, where* C *is a constant.*

So, if the derivative of a function is zero in an interval, then the function is constant in that interval.

Corollary 1

Proof.

Suppose $f'(x) = 0$ at each point x of an open interval (a, b). Let $a \leq x < y \leq b$.

Then f is continuous on the closed interval $[x, y]$ and differentiable on in the open interval (x, y).

By the Mean Value Theorem, there exists c, $x < c < y$, with

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

However, $f'(c) = 0$, so $f(x) = f(y)$. Since x, y are arbitrary in the interval $[a, b]$, f is constant on the interval $[a, b]$. □

Corollary 2

Another important result of the Mean Value Theorem which follows immediately from the last one is this:

Corollary

If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is constant on (a, b).

So, if the two functions have the same derivative then those functions differ by a constant.

Corollary 2

Proof.

Suppose $f'(x) = g'(x)$ at each point x of an open interval (a, b).

Let $h = f - g$. Then h is differentiable on (a, b) and $h'(x) = 0$ for all x in (a, b).

By Corollary 1, h is constant. It follows that $f(x) - g(x)$ is constant on (a, b). □

Corollary 3

We know that if f is differentiable and increasing on an interval (a, b), then $f'(x) \geq 0$ for all x in (a, b).

Similarly, if f is differentiable and decreasing on an interval (a, b), then $f'(x) \leq 0$ for all x in (a, b).

Corollary 3

Another important result of the Mean Value Theorem is the converse of this result.

Corollary

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

- i *If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.*
- ii *If $f'(x) < 0$ for all $x \in (a, b)$, then f is an decreasing function over $[a, b]$.*

Corollary 3

Proof.

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Suppose $f'(x) > 0$ for all $x \in (a, b)$.

Let $a \leq x < y \leq b$. Then f is continuous on the closed interval closed interval $[x, y]$ and differentiable over the open interval (x, y). By the Mean Value Theorem, there exists $c \in (x, y)$ so that

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

Since $x < y$ and $f'(c) > 0$, this forces $f(x) < f(y)$. Since x, y with $a \leq x < y \leq b$ are arbitrary, f is an increasing function over $[a, b]$.

The second statement is proved completely analogously. □