Antiderivatives

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Outline

Antiderivatives

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So far, we have been starting with a function $f(x)$, taking its derivative, and producing $f'(x)$.

Now, we reverse the process. We will be given $f'(x)$ and produce the function $f(x)$. The function $f(x)$ is the **antiderivative** of $f'(x)$.

Notation for Antiderivatives

Notation for Antiderivatives

If we are given a function $f(x)$, its antiderivative is denoted

$$F(x) = \int f(x)\,dx$$

or

$$F(x) = \int f.$$

The antiderivative is also called the **indefinite integral**.

Constant of Integration

Constant of Integration

Recall from the section the Mean Value Theorem and its corollary that if $f'(x) = g'(x)$, then $f(x) = g(x) + C$, for some constant C. This means that a given function doesn't have one antiderivative, but has a family of antiderivatives, all differing by a constant.

For example, the derivative of x^2 is $2x$. But the derivative of $x^2 + 3$, $x^2 - 6$,$x^2 + 47$, $x^2 + e - \pi$ is also $2x$.

We write this as

$$\int 2x\,dx = x^2 + C.$$

The C here is called the **constant of integration**.

Examples

Examples

- $\int 1\,dx = x + C$ because $(x)' = 1$.
- $\int 2x\,dx = x^2 + C$ because $(x^2)' = 2x$.
- $\int \cos x\,dx = \sin x + C$ because $(\sin x)' = \cos x$.
- $\int 2^x \ln 2\,dx = 2^x + C$ because $(2^x)' = 2^x \ln 2$.
- $\int \frac{1}{1+x^2}\,dx = \arctan x + C$ because $(\arctan x)' = \frac{1}{1+x^2}$.

An Important Comment

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Notice this means you need to be able to recognize what is a derivative.

That means you have to know all the derivatives of the basic functions!

Basic Antidifferentiation Formulas

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$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int k \, dx = kx + C, \text{ for } k \text{ a constant}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \text{ for } n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

Basic Antidifferentiation Formulas

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int -\frac{1}{1+x^2}\,dx = \operatorname{arccot} x + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C$$

$$\int -\frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arccsc} x + C.$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \text{ for } a > 0, a \neq 1$$

$$\int e^x\,dx = e^x + C,$$

Properties of Indefinite Integrals

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectivley, and let k be any real number.

Then
$$\int f(x) \pm g(x)\, dx = F(x) \pm G(x) + C$$
and
$$\int kf(x)\, dx = kF(x) + C.$$

Initial Value Problems and Differential Equations

Initial Value Problems and Differential Equations

An equation of the form $dy/dx = f(x)$ is called a **differential equation**. The goal here is to find the function y explicitly as a function of x.

If you're also given an initial condition $y(x_0) = y_0$, then the differential equation together with the initial condition is called an **initial value problem**.

You solve a differential equation of this form by taking the antiderivative. The initial condition then tells you the value of the constant of integration.

Example

Example 1

Example

Solve the initial value problem

$$\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0.$$

Example 1

Solution

We solve this by integrating both sides of the equation $dy/dx = 9x^2 - 4x + 5$*.*

$$\begin{aligned} y &= \int 9x^2 - 4x + 5\,dx \\ &= 3x^3 - 2x^2 + 5x + C. \end{aligned}$$

Example 1

Solution

We substitute the initial value to find the value of C.

$$\begin{aligned} y &= 3x^3 - 2x^2 + 5x + C \\ 0 &= 3(-1)^3 - 2(-1)^2 + 5(-1) + C \\ &= -3 - 2 - 5 + C \\ C &= 10. \end{aligned}$$

So, the solution to the initial value problem is

$$y = 3x^3 - 2x^2 + 5x + 10.$$

Indefinite Integrals

Indefinite Integrals

Definition

The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Examples

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$$\int (5 - 6x)\,dx = 5x - 3x^2 + C.$$

$$\int x^{-5/4} = \frac{x^{-1/4}}{-1/4} + C = -4x^{-1/4} + C.$$

$$\begin{aligned}\int (\sqrt{x} + \sqrt[3]{x})\,dx &= \int (x^{1/2} + x^{1/3})\,dx \\ &= \frac{x^{3/2}}{3/2} + \frac{x^{4/3}}{4/3} + C \\ &= \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C.\end{aligned}$$