Derivatives of Exponential and Logarithmic Functions

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Outline

The Exponential Rule

The Exponential Rule

When we apply the definition of derivative to the function $f(x) = a^x$ for $a > 0$, $a \neq 1$, we get

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\ &= \lim_{h\to 0} a^x \cdot \frac{a^h - 1}{h} = a^x \left(\lim_{h\to 0} \frac{a^h - 1}{h} \right).\end{aligned}$$

So, the derivative of a^x is some constant times a^x.

We note that this constant is $f'(0)$.

Derivatives of Exponential Functions

We examine the limit

$$\lim_{h \to 0} \frac{a^h - 1}{h}.$$

For $a = 2$, this limit is approximately 0.69.

For $a = 3$, this limit is approximately 1.10.

It makes sense that for some value between 2 and 3, this limit is 1. We define e to be this value. So,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Derivatives of Exponential Functions

By our choice of e, we have the following:

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x.$$

Derivative of the Natural Logarithm Function

Derivative of the Natural Logarithm Function

Suppose $y = \ln(x)$. Since the natural logarithm and the exponential functions are inverse functions, we have

$$x = e^y.$$

We take the derivative of this implicitly, treating y as a function of x. This gives us

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}e^y \\ 1 &= e^y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. \end{aligned}$$

Derivative of the Natural Logarithm Function

This gives us the following important result:

Derivative of the $\ln(x)$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0.$$

Derivative of the Natural Logarithm Function

We can extend this result using the Chain Rule. If u is a differentiable function of x, then

Derivative of the $\ln(u)$

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}, \quad u > 0.$$

Examples

Example 3

Example

Find dy/dx.

1. $y = \dfrac{1}{\ln 3x}$
2. $y = \ln(\sin x)$
3. $y = x \ln \sqrt{x}$

Example 3

Solution

1. $$\frac{dy}{dx} = \frac{0 \cdot \ln 3x - 1 \cdot \frac{1}{3x} \cdot 3}{(\ln 3x)^2} = \frac{-1}{x(\ln 3x)^2}.$$
2. $$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x.$$
3. $$\frac{dy}{dx} = (1) \ln \sqrt{x} + x \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2} = \ln(\sqrt{x}) + \frac{1}{2}.$$

Example 4

Example

Find the derivative of

$$y = (x^2 \ln x)^4.$$

Example 4

Solution

We take the derivative using the Power Rule, the Chain Rule, and the Product Rule.

$$\begin{aligned}\frac{dy}{dx} &= 4(x^2 \ln x)^3 \cdot \left[2x \ln x + x^2 \cdot \frac{1}{x}\right] \\ &= 4(x^2 \ln x)^3 \left(2x \ln x + x\right) \\ &= 4x(x^2 \ln x)^3 \left(2 \ln x + 1\right).\end{aligned}$$

Derivative of a^u and $\log_a(u)$

Derivative of a^u and $\log_a u$

Suppose $y = a^x$. Then $\ln(y) = \ln(a^x) = x \ln a$, by the Power Rule for logarithms. Raising e to this power gives

$$y = e^{\ln y} = e^{x \ln a}.$$

Taking the derivative using the Chain Rule, we get

$$\begin{aligned} \frac{dy}{dx} &= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \\ &= e^{x \ln a} \cdot \ln a \\ &= a^x \ln a. \end{aligned}$$

Derivative of a^u and $\log_a(u)$

This gives us the following important result:

Derivative of a^x

$$\frac{d}{dx}a^x = a^x \ln a, \quad x > 0.$$

Derivative of a^u and $\log_a(u)$

We can extend this result using the Chain Rule:

Derivative of a^u

If u is a differentiable function of x, then a^u is a differentiable function of x wherever $u > 0$, and

$$\frac{d}{dx}a^u = a^u \ln a \cdot \frac{du}{dx}.$$

Derivative of a^u and $\log_a u$

Suppose $y = \log_a x$. Then $a^y = x$ since $\log_a x$ and a^x are inverse functions. Taking the derivative of $a^y = x$, treating y as a differentiable function of x, we get

$$\begin{aligned} \frac{d}{dx} a^y &= \frac{d}{dx} x \\ a^y \ln a \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{a^y \ln a} \\ &= \frac{1}{x \ln a}. \end{aligned}$$

Derivative of a^u and $\log_a(u)$

This gives us the following important result:

Derivative of $\log_a x$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}, \quad x > 0.$$

Derivative of a^u and $\log_a u$

We can extend this result using the Chain Rule:

Derivative of $\log_a u$

If u is a differentiable function of x, then $\log_a u$ is a differentiable function of x wherever $u > 0$, and

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \cdot \frac{du}{dx}$$

Example

Example 5

Example

Find dy/dx.

1. $y = 2^x$
2. $y = \log_4 x + \log_4 x^2$
3. $y = 3^{\log_2 x}$

Solution

1. $dy/dx = 2^x \ln 2$
2. $dy/dx = \frac{1}{x \ln 4} + \frac{1}{x^2 \ln 4} \cdot 2x = \frac{3}{x \ln 4}$
3. $dy/dx = 3^{\log_2 x} \ln 3 \cdot \dfrac{1}{x \ln 2} = \dfrac{3^{\log_2 x} \ln 3}{x \ln 2}$

Logarithmic Differentiation

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**.

Example

Example 6

Example

Use logarithmic differentiation to find the derivative of dy/dx if $y = \sqrt{(x^2+1)(x-1)^2}$.

Solution

We start by taking the natural logarithm of both sides of the equation and using the rules of logarithms:

$$\begin{aligned}
\ln y = \ln\left(\sqrt{(x^2+1)(x-1)^2}\right) &= \frac{1}{2}\ln\left[(x^2+1)(x-1)^2\right] \\
&= \frac{1}{2}\left[\ln(x^2+1) + \ln\left((x-1)^2\right)\right] \\
&= \frac{1}{2}\left[\ln(x^2+1) + 2\ln(x-1)\right] \\
&= \frac{1}{2}\ln(x^2+1) + \ln(x-1).
\end{aligned}$$

Example 6

Solution

Now, take the derivative implicitly with respect to x:

$$\begin{aligned}
\ln y &= \frac{1}{2}\ln(x^2+1) + \ln(x-1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot(2x) + \frac{1}{x-1} \\
&= \frac{x}{x^2+1} + \frac{1}{x-1}.
\end{aligned}$$

Example 6

Solution

Finally, multiply both sides by y to solve for $\frac{dy}{dx}$.

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{x}{x^2+1} + \frac{1}{x-1} \\ \frac{dy}{dx} &= y\left[\frac{x}{x^2+1} + \frac{1}{x-1}\right] \\ &= \sqrt{(x^2+1)(x-1)^2}\left[\frac{x}{x^2+1} + \frac{1}{x-1}\right] \\ &= \frac{x(x-1)}{\sqrt{x^2+1}} + \sqrt{x^2+1}.\end{aligned}$$

Irrational Exponents and the Power Rule (General Version)

Irrational Exponents and the Power Rule (General Version)

If we want to define what it means to take a real number $x > 0$ to any real power n, we certainly want $\ln(x^n) = n \ln x$. This motivates the following definition.

Definition

For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

General Power Rule for Derivatives

General Power Rule for Derivatives

For any $x > 0$ and for any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} exist.

General Power Rule for Derivatives

Proof.

If n is any real number, then $x^n = e^{n \ln x}$, by definition. Taking the derivative with respect to x, we get

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} \\ &= e^{n \ln x} \cdot \frac{n}{x} \\ &= x^n \cdot \frac{n}{x} \\ &= n x^{n-1}. \end{aligned}$$

□

Example

Example 7

Example

Use logarithmic differentiation to find the derivative of
$y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$.

Solution

We start by taking the natural logarithm of both sides of the equation and using the rules fo logarithms:

$$\begin{aligned}\ln y &= \ln\left(\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\right)\\ &= \ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1).\end{aligned}$$

Example 7

Solution

Now, take the derivative implicitly with respect to x:

$$\begin{aligned}\ln y &= \ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{2}{3}\cdot\frac{1}{x+1} \\ &= \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\end{aligned}$$

Example 7

Solution

Finally, multiply both sides by y to solve for $\frac{dy}{dx}$.

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\\ \frac{dy}{dx} &= y\left[\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right]\\ &= \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right]\\ &= \frac{\sqrt{x^2+1}}{(x+1)^{2/3}}+\frac{x^2}{(x+1)^{2/3}\sqrt{x^2+1}}-\frac{2x\sqrt{x^2+1}}{3(x+1)^{5/3}}.\end{aligned}$$

The Number e Expressed as a Limit

The Number e Expressed as a Limit

Earlier in the course, we defined e to be the number so that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

We also noted then that e is between 2 and 3.

The Number e Expressed as a Limit

Theorem

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof.

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$, so $f'(1) = 1$. By the definition of the derivative

$$\begin{aligned} f'(1) &= \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} \\ &= \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \to 0} \ln[(1+x)^{1/x}]. \end{aligned}$$

□

The Number e Expressed as a Limit

Proof.

But we know that $f'(1) = 1$, so

$$\lim_{x\to 0} \ln[(1+x)^{1/x}] = 1$$

$$\lim_{x\to 0} e^{\ln[(1+x)^{1/x}]} = e^1$$

$$\lim_{x\to 0} (1+x)^{1/x} = e.$$

□