Implicit Differentiation

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Outline

Implicitly Defined Functions

Implicitly Defined Functions

Generally our functions are given as $y = f(x)$. The dependent variable y is given explicitly as a function of the independent variable x. However, it is also possible to define functions implicitly. As an example, take the equation

$$x^2 + y^2 = 25.$$

This equation can define y as a function of x (actually two functions of x) :

$$y = \pm\sqrt{25 - x^2},$$

Let's take the derivative of each of these two functions.

Implicitly Defined Functions

For

$$y = \sqrt{25 - x^2},$$

we have

$$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}.$$

Notice this can be written as

$$\frac{dy}{dx} = \frac{-x}{y}.$$

Implicitly Defined Functions

For

$$y = -\sqrt{25 - x^2},$$

we have

$$\frac{dy}{dx} = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}}.$$

Notice this can also be written as

$$\frac{dy}{dx} = \frac{-x}{y}.$$

Implicitly Defined Functions

For implicitly defined functions such as

$$x^2 + y^2 = 25,$$

we would like to be able to compute $\frac{dy}{dx}$ directly without solving the equation for y.

Implicitly Defined Functions

What you do here is (1) identify which letter is the variable, (2) identify which letter is the function, and (3) take the derivative of the entire equation with respect to the variable—remembering to use the Chain Rule.

For instance, for

$$x^2 + y^2 = 25,$$

the variable is x and y is a function of x. We take the derivative of the equation remembering that x is the variable and y is a function of x.

Implicitly Defined Functions

We carry out this plan of attack in this example:

$$\begin{aligned} x^2 + y^2 &= 25 \\ 2x\frac{dx}{dx} + 2y\frac{dy}{dx} &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y}. \end{aligned}$$

Notice this gives you the derivative of both functions defined implicitly by this equation.

Examples

Example 1

Example

Suppose the equation $x^2y + xy^2 = 6$ defines y as a differentiable function of x. Compute dy/dx.

Example 1

Solution

The crucial thing to remember here is that there is only one variable.

The remaining letters are either constants or mysterious functions of the variable. In this case, x *is the variable and* y *is some function of* x*.*

Remembering that, we take the derivative of the entire equation.

Example 1

Solution

This method of attack gives us

$$\left[\frac{d}{dx}(x^2)\cdot y + x^2\cdot\frac{d}{dx}(y)\right] + \left[\frac{d}{dx}(x)\cdot y^2 + x\cdot\frac{d}{dx}(y^2)\right] = \frac{d}{dx}(6)$$

$$\left[2x\cdot y + x^2\cdot\frac{dy}{dx}\right] + \left[(1)\cdot y^2 + x\cdot 2y\cdot\frac{d}{dx}(y)\right] = 0$$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0.$$

Example 1

Solution

Next, you solve this equation for $\frac{dy}{dx}$.

$$\begin{aligned} 2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(x^2 + 2xy\right) &= -2xy - y^2 \\ \frac{dy}{dx} &= \frac{-2xy - y^2}{x^2 + 2xy} \\ &= -\frac{y(2x+y)}{x(x+2y)}. \end{aligned}$$

Example 2

Example

Find the slope of the tangent line to the curve

$$x^2 + xy - y^2 = 1$$

at the point $(2, 3)$.

Example 1

Solution

Recall $x^2 + xy - y^2 = 1.$

First, we compute the derivative implicitly:

$$2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0.$$

Example 2

Solution

Now we substitute the values we're given and solve for $\frac{dy}{dx}$.

$$\begin{aligned} 2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} &= 0 \\ 2(2) + (3) + 2\frac{dy}{dx} - 2(3)\frac{dy}{dx} &= 0 \\ 7 - 4\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{7}{4} \end{aligned}$$

at the point $(2,3)$. *This is the slope of the tangent line.*

Implicit Differentiation

Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example

Example 3

Example

Compute dy/dx if

$$e^{2x} = \sin(x + 3y).$$

Example 1

Solution

Recall $e^{2x} = \sin(x + 3y)$.

First we take the derivative treating y *as a function of the variable* x:

$$2e^{2x} = \cos(x + 3y)\left(1 + 3\frac{dy}{dx}\right)$$

$$2e^{2x} = \cos(x + 3y) + 3\cos(x + 3y)\frac{dy}{dx}.$$

Example 3

Solution

Now solve for dy/dx:

$$\begin{aligned} 2e^{2x} &= \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx} \\ 2e^{2x} - \cos(x+3y) &= 3\cos(x+3y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}. \end{aligned}$$

Derivatives of Higher Order

Derivatives of Higher Order

To find the second derivative implicitly, follow the following procedure:

1. Compute dy/dx. This gives you the first derivative in terms of x and y.
2. Take the next derivative. This gives d^2y/dx^2.
3. If necessary, substitute the expression for dy/dx obtained from step 1 into the expression for d^2y/dx^2 obtained in the last step.
4. Simplify the expression for d^2y/dx^2 algebraically.

Examples

Example 4

Example

Compute d^2y/dx^2 if
$$xy + y^2 = 1.$$

Example 4

Solution

Recall that $xy + y^2 = 1$.

We compute the first derivative implicitly:

$$\begin{aligned} y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-y}{x+2y}. \end{aligned}$$

Example 4

Solution

Now, take the next derivative using the Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-y}{x+2y} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}(x+2y) - (-y)(1+2\frac{dy}{dx})}{(x+2y)^2} \\
&= \frac{-\frac{dy}{dx}(x+2y) + y(1+2\frac{dy}{dx})}{(x+2y)^2} \\
&= \frac{-x\frac{dy}{dx} - 2y\frac{dy}{dx} + y + 2y\frac{dy}{dx}}{(x+2y)^2} = \frac{-x\frac{dy}{dx} + y}{(x+2y)^2}.
\end{aligned}$$

Example 4

Solution

Finally, we substitute $\frac{dy}{dx}$ *into the expression for* $\frac{d^2y}{dx^2}$ *and simplify.*

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{-x\frac{dy}{dx} + y}{(x+2y)^2} \\
&= \frac{-x\left(\frac{-y}{x+2y}\right) + y}{(x+2y)^2} = \frac{\left(\frac{xy}{x+2y}\right) + y}{(x+2y)^2} \\
&= \frac{xy + y(x+2y)}{(x+2y)^3} \\
&= \frac{2y(x+y)}{(x+2y)^3}.
\end{aligned}$$

Example 5

Example

Find the lines that are tangent and normal to the curve

$$x^2 + y^2 = 25$$

at the point $(3, -4)$.

Example 5

Solution

First, we compute dy/dx implicitly.

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}. \end{aligned}$$

At the point $(3, -4)$, $dy/dx = 3/4$. This is the slope of the tangent line. The slope of the normal line is the negative reciprocal of this: $-4/3$.

Example 5

Solution

So, the tangent line to the curve $x^2 + y^2 = 25$ *at the point* $(3, -4)$ *is*

$$y - (-4) = \frac{3}{4}(x - 3)$$
$$y = \frac{3}{4}x - \frac{25}{4}$$

Example 5

Solution

And the normal line to the curve $x^2 + y^2 = 25$ *at the point* $(3, -4)$ *is*

$$y - (-4) = -\frac{4}{3}(x - 3)$$
$$y = -\frac{4}{3}x.$$