Derivatives of Inverse Functions

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Outline

Derivatives of Inverses of Differentiable Functions

Derivatives of Inverses of Differentiable Functions

Recall that a function f has an inverse function g if $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f.

Derivatives of Inverses of Differentiable Functions

For example, $f(x) = 2x + 4$ and $g(x) = \frac{1}{2}x - 2$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}x - 2\right) = \left(2\left(\frac{1}{2}x - 2\right) + 4\right) = x.$$

$$(g \circ f)(x) = g(f(x)) = g\left(2x + 4\right) = \frac{1}{2}\left(2x + 4\right) - 2 = x.$$

So, f and g are inverse functions.

Notice that $\frac{df}{dx} = 2$ and $\frac{dg}{dx} = \frac{1}{2}$.

Derivatives of Inverses of Differentiable Functions

For another example, $f(x) = x^3 - 2$ and $g(x) = \sqrt[3]{x+2}$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x+2}\right) = \left(\sqrt[3]{x+2}\right)^3 - 2 = x.$$

$$(g \circ f)(x) = g(f(x)) = g\left(x^3 - 2\right) = \sqrt[3]{(x^3-2)+2} = x.$$

So, f and g are inverse functions.

Notice that $\frac{df}{dx} = 3x^2$ and $\frac{dg}{dx} = \frac{1}{3}(x+2)^{-2/3} = \frac{1}{3(x+2)^{2/3}}$, which is $\frac{1}{3y^2}$ if $y = \sqrt[3]{x+2}$.

Derivatives of Inverses of Differentiable Functions

Let's examine this more closely. Suppose f and g are inverse functions. Then

$$(f \circ g)(x) = f(g(x)). \tag{1}$$

Let $y = g(x)$. Then $x = f(y)$. We take the derivative of Equation (1) and get

$$\begin{aligned} f'(g(x)) \cdot g'(x) &= 1 \\ f'(y) \cdot g'(x) &= 1 \\ g'(x) &= \frac{1}{f'(y)} = \frac{1}{f'(g(x))}. \end{aligned}$$

This is the relationship between the derivative of a function f and the derivative of its inverse function g.

Derivatives of Inverses of Differentiable Functions

Theorem

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (which is the range of f). The value of $(f^{-1})'$ at a point x, where $x = f(y)$, in the domain of f^{-1} is the reciprocal of the value of $f'(y) = f'(f^{-1}(x))$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Examples

Example 1

Example

The function $f(x) = x^3$ and its inverse function $f^{-1}(x) = \sqrt[3]{x}$ derivatives $f'(x) = 3x^2$ and $(f^{-1})'(x) = \left(\frac{1}{3}x^{-2/3}\right)$. Let's verify the preceding theorem for these functions.

Example 1

Solution

By the theorem,

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{f'(\sqrt[3]{x})} \\ &= \frac{1}{3(\sqrt[3]{x})^2} \\ &= \frac{1}{3x^{2/3}}.\end{aligned}$$

Example 2

Example

The function $f(x) = x^2 - 1$, $x \geq 0$, and its inverse function $f^{-1}(x) = \sqrt{x+1}$ derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = \frac{1}{2}(x+1)^{-1/2}$. Let's verify the preceding theorem for these functions.

Example 2

Solution

By the theorem,

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{f'(\sqrt{x+1})}\\ &= \frac{1}{2\sqrt{x+1}}.\end{aligned}$$

Extending the Power Rule to Rational Exponents

Extending the Power Rule to Rational Exponents

Theorem (Extending the Power Rule to Rational Exponents)

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$

Extending the Power Rule to Rational Exponents

Proof.

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = nx^{n-1} \text{ and } f'(g(x)) = n\left(x^{1/n}\right)^{n-1} = nx^{1-1/n}.$$

Finally

$$g'(x) = \frac{1}{nx^{1-1/n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

Extending the Power Rule to Rational Exponents

Proof.

To differentiate $x^{m/n}$ we must rewrite is as $\left(x^{1/n}\right)^m$ and apply the chain rule. Thus,

$$\begin{aligned} \frac{d}{dx}x^{m/n} &= \frac{d}{dx}\left(\left(x^{1/n}\right)^m\right) \\ &= m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n}x^{\frac{1}{n}-1} \\ &= \frac{m}{n}x^{\frac{m}{n}-1}. \end{aligned}$$

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Derivative of the Inverse Sine Function

Example

Use the inverse function theorem to find the derivative of

$$g(x) = \sin^{-1} x.$$

Derivative of the Inverse Sine Function

Solution

Since $g(x) = \sin^{-1} x$, *we have* $f(x) = \sin x$. *Applying the formula for derivatives of inverse functions, we get*

$$
\begin{aligned}
g'(x) &= \frac{1}{f'(g(x))} \\
&= \frac{1}{\cos(\sin^{-1} x)}
\end{aligned}
$$

Derivative of the Inverse Sine Function

Solution

Since $y = \sin^{-1} x$ *satisfies* $-\pi/2 \leq y \leq \pi/2$, *we have the* $\cos y \geq 0$. *Hence*

$$\cos\left(\sin^{-1} x\right) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

So, we have

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\cos\left(\sin^{-1} x\right)} = \frac{1}{\sqrt{1 - x^2}}.$$

Derivative of the Inverse Trigonometric Functions

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Theorem (Derivatives of Inverse Trigonometric Functions)

- $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx} \cot^{-1} x = \dfrac{-1}{1+x^2}$
- $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2-1}}$
- $\dfrac{d}{dx} \csc^{-1} x = \dfrac{-1}{|x|\sqrt{x^2-1}}$