The Chain Rule

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Outline

Derivative of a Composite Function

Derivative of a Composite Function

Consider the function $y = (3x + 1)^2$. We compute its derivative using the Product Rule:

$$\begin{aligned} y &= (3x+1)^2 \\ &= (3x+1)(3x+1) \\ \frac{dy}{dx} &= \frac{d}{dx}(3x+1)\cdot(3x+1) + (3x+1)\cdot\frac{d}{dx}(3x+1) \\ &= (3)\cdot(3x+1) + (3x+1)\cdot(3) \\ &= 6(3x+1). \end{aligned}$$

Notice that the derivative is **not** $2(3x + 1)$. There is something else happening here.

Derivative of a Composite Function

Consider the function $y = \sin^2 x$. We compute its derivative using the Product Rule:

$$\begin{aligned} y &= \sin^2 x \\ &= \sin x \cdot \sin x \\ \frac{dy}{dx} &= \frac{d}{dx}(\sin x) \cdot (\sin x) + (\sin x) \cdot \frac{d}{dx}(\sin x) \\ &= (\cos x) \cdot (\sin x) + (\sin x) \cdot (\cos x) \\ &= 2 \sin x \cos x. \end{aligned}$$

Notice that the derivative is **not** $2 \sin x$. There is something else happening here.

Derivative of a Composite Function

Let's do a little thought experiment. Recall that the derivative is a rate of change.

Suppose your car gets 30 miles/gallon. Suppose gasoline is \$2.00/gallon. If you want to compute the cost of operating your car per mile (rather than cost per gallon), you do what chemists call dimensional analysis.

Derivative of a Composite Function

We get

$$15 \text{ miles/dollar} = \frac{30 \text{ miles/gallon}}{2 \text{ dollars/gallon}}.$$

Or, if you look at it in a different way,

$$30 \ \frac{\text{miles}}{\text{gallon}} = 15 \ \frac{\text{miles}}{\text{dollar}} \cdot 2 \ \frac{\text{dollars}}{\text{gallon}}.$$

You multiply the rates together.

Derivative of a Composite Function

The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, in Leibniz's notation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Proof of Chain Rule

Proof of Chain Rule

Proof: Suppose $f(u)$ is a differentiable function of u and $u(x)$ is a differentiable function of x.

Since $f(u)$ is a differentiable function of u,

$$\lim_{\Delta u \to 0} \frac{f(u + \Delta u) - f(u)}{\Delta u} = f'(u).$$

Let

$$\epsilon_1 = \frac{f(u + \Delta u) - f(u)}{\Delta u} - f'(u).$$

Then ϵ_1 goes to zero at $\Delta u \to 0$ and

$$f(u + \Delta u) - f(u) = (f'(u) + \epsilon_1)\Delta u.$$

Proof of Chain Rule

Since $u(x)$ is a differentiable function of x,

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} = u'(x).$$

Let

$$\epsilon_2 = \frac{u(x+\Delta x) - u(x)}{\Delta x} - u'(x).$$

Then ϵ_2 goes to zero at $\Delta x \to 0$ and

$$u(x+\Delta x) - u(x) = (u'(x) + \epsilon_2)\Delta x.$$

Proof of Chain Rule

So, we have

$$f(u+\Delta u) - f(u) = (f'(u)+\epsilon_1)\Delta u \tag{1}$$
$$u(x+\Delta x) - u(x) = (u'(x)+\epsilon_2)\Delta x \tag{2}$$

Let $\Delta u = u(x+\Delta x) - u(x)$. Since u is differentiable at x, it is continuous at x. So as Δx goes to zero, Δu goes to zero as well.

Substituting this into Equation 1 and using Equation 2, we get

$$\begin{aligned} f(u(x+\Delta x)) - f(u(x)) &= (f'(u(x))+\epsilon_1)\Delta u \\ &= (f'(u(x))+\epsilon_1)(u'(x)+\epsilon_2)\Delta x. \end{aligned}$$

Proof of Chain Rule

From the preceding slide, we have

$$f(u(x+\Delta x)) - f(u(x)) = (f'(u(x)) + \epsilon_1)(u'(x) + \epsilon_2)\Delta x.$$

Dividing by Δx and letting $\Delta x \to 0$, we get

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{f(u(x+\Delta x)) - f(u(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(f'(u(x)) + \epsilon_1)(u'(x) + \epsilon_2)\right] \\
&= \lim_{\Delta x \to 0} \left[(f'(u(x)) + \epsilon_1)\right] \lim_{\Delta x \to 0} \left[(u'(x) + \epsilon_2)\right] \\
&= \lim_{\Delta u \to 0} \left[(f'(u(x)) + \epsilon_1)\right] \lim_{\Delta x \to 0} \left[(u'(x) + \epsilon_2)\right] \\
&= f'(u(x))u'(x),
\end{aligned}$$

since $\epsilon_1 \to 0$ as $\Delta u \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$. $\square$

The Chain Rule Using Leibniz's Notation

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This makes the chain rule easy to remember and shows you why Leibniz's notation is so useful.

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Examples

Example 1

Example

Compute the derivative of $y = 2(8x - 1)^3$.

Example 1

Solution

Let $u = 8x - 1$. *This is the inside function. Let* $y = 2u^3$. *This is the outside function. By the Chain Rule*

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 6u^2 \cdot 8 \\ &= 48u^2 \\ &= 48(8x-1)^2.\end{aligned}$$

As x *changes, the rate at which* y *changes is the rate at which* y *changes with respect to* u *times the rate at which* u *changes with respect to* x.

Example 2

Example

Compute the derivative of $y = \tan^3 x$.

Example 2

Solution

Let $u = \tan x$*. This is the inside function. Let* $y = u^3$*. This is the outside function. By the Chain Rule*

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 3u^2 \cdot \sec^2 x \\ &= 3(\tan x)^2 \cdot \sec^2 x \\ &= 3\tan^2 x \sec^2 x.\end{aligned}$$

“Outside-Inside” Rule

“Outside-Inside” Rule

If you write the Chain Rule this way

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x),$$

we can talk about the Chain Rule in these terms:

You take the derivative of the outside function f leaving the inside function alone, and then multiply by the derivative of the inside function. This is multiplying the rates together.

Examples

Example 3

Example

Compute the derivative of

$$y = (4 - 3x)^9.$$

Example 3

Solution

Here, the inside function is $u = 4 - 3x$ *and the outside function is* $y = u^9$. *The derivative of the outside function is* $\frac{dy}{du} = 9u^8$. *The derivative of the inside function is* $\frac{du}{dx} = -3$. *The derivative is then*

$$\begin{aligned}\frac{dy}{dx} &= 9u^8 \cdot (-3) \\ &= 9(4-3x)^8 \cdot (-3) \\ &= -27(4-3x)^8.\end{aligned}$$

Example 4

Example

Compute the derivative of

$$y = \sqrt[3]{2x - x^2}.$$

Example 4

Solution

Here, the inside function is $u = 2x - x^2$ *and the outside function is* $y = \sqrt[3]{u} = u^{1/3}$. *The derivative of the outside function is* $\frac{dy}{du} = \frac{1}{3}u^{-2/3}$. *The derivative of the inside function is* $\frac{du}{dx} = 2 - 2x$. *The derivative is then*

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{3}u^{-2/3} \cdot (2 - 2x) \\ &= \frac{1}{3}(2x - x^2)^{-2/3} \cdot (2 - 2x) \\ &= \frac{2 - 2x}{3(2x - x^2)^{2/3}}. \end{aligned}$$

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x))g'(x)$.

Problem-Solving Strategy: Applying the Chain Rule

Remark

When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in.

It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on.

Also, remember that we never evaluate a derivative at a derivative.

Repeated Use of the Chain Rule

Repeated Use of the Chain Rule

The Chain Rule must be used with each composed function. So, if several functions are nested, you must use the Chain Rule each time there is a new function.

Example

Example 5

Example

Find dy/dt if
$$y = \sin^2(3t - 2).$$

Example 5

Solution

Here we have three composed functions.

The inside function is $x = 3t - 2$. *The middle function is* $u = \sin(x)$ *The outside function is* $y = u^2$.

By the Chain Rule, the derivative is

$$\begin{aligned}\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt} &= 2u \cdot \cos(x) \cdot (3) \\ &= 2\sin(x) \cdot \cos(x) \cdot (3) \\ &= 2\sin(3t-2) \cdot \cos(3t-2) \cdot (3) \\ &= 6\sin(3t-2)\cos(3t-2).\end{aligned}$$

The Chain Rule with Powers of a Function

The Chain Rule with Powers of a Function

If n is any real number and f is a power function $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is then some differentiable function of x, the the Chain Rule tells us

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Your text calls this the Power Chain Rule.

Example

Example 6

Example

Find dy/dx if

$$y = \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^4 .$$

Example 6

Solution

By the Power Chain Rule, we have

$$\begin{aligned}
\frac{dy}{dx} &= 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \cdot \frac{d}{dx}\left(\frac{1}{8}x^2 + x - x^{-1}\right) \\
&= 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \cdot \left(\frac{1}{4}x + 1 - (-x^{-2})\right) \\
&= 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \cdot \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)
\end{aligned}$$

Combining the Chain Rule with Other Rules

Of course, all the rules you already know are applied when needed. So you may need the Product Rule, Quotient Rule, and Chain Rule in many combinations depending on the particular problem.

Examples

Example 7

Example

Find the dy/dx if $y = xe^{2x}$.

Example 7

Solution

This is the product of two functions, so we apply the Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}x \cdot e^{2x} + x \cdot \frac{d}{dx}e^{2x}$$

To find the derivative of e^{2x} *we must use the Chain Rule:*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}x \cdot e^{2x} + x \cdot \frac{d}{dx}e^{2x} \\ &= (1) \cdot e^{2x} + x \cdot e^{2x}\frac{d}{dx}(2x) \\ &= (1) \cdot e^{2x} + x \cdot e^{2x} \cdot 2 \\ &= e^{2x} + 2xe^{2x} = e^{2x}(1 + 2x). \end{aligned}$$

Example 8

Example

Find the dy/dx if $y = (2x-5)^{-1}(x^2-5x)^6$.

Example 8

Solution

This is the product of two functions, so we use the Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}[(2x-5)^{-1}](x^2-5x)^6 + (2x-5)^{-1}\frac{d}{dx}[(x^2-5x)^6]$$

To take these two derivatives, we must use the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(2x-5)^{-1}](x^2-5x)^6 + (2x-5)^{-1}\frac{d}{dx}[(x^2-5x)^6] \\ &= (-1)(2x-5)^{-2}\cdot\frac{d}{dx}(2x-5)\cdot(x^2-5x)^6 \\ &\quad + (2x-5)^{-1}\cdot 6(x^2-5x)^5\cdot\frac{d}{dx}(x^2-5x)\end{aligned}$$

Example 8

Solution

Continuing, we get

$$\begin{aligned}\frac{dy}{dx} &= (-1)(2x-5)^{-2}\cdot\frac{d}{dx}(2x-5)\cdot(x^2-5x)^6\\ &\qquad + (2x-5)^{-1}\cdot 6(x^2-5x)^5\cdot\frac{d}{dx}(x^2-5x)\\ &= (-1)(2x-5)^{-2}\cdot(2)\cdot(x^2-5x)^6\\ &\qquad + (2x-5)^{-1}\cdot 6(x^2-5x)^5\cdot\ (2x-5)\\ &= -2(2x-5)^{-2}(x^2-5x)^6 + 6(x^2-5x)^5\end{aligned}$$

Example 8

Solution

Continuing, we get

$$\begin{aligned}\frac{dy}{dx} &= -2(2x-5)^{-2}(x^2-5x)^6 + 6(x^2-5x)^5 \\ &= -\frac{2(x^2-5x)^5}{(2x-5)^2}\left[(x^2-5x) - 3(2x-5)^2\right] \\ &= \frac{2(x^2-5x)^5}{(2x-5)^2}(11x^2-55x+75).\end{aligned}$$

Example 9

Example

Find the dy/dx if $y = \tan\left(\frac{\sin x}{x}\right)$.

Example 9

Solution

This is first the composition of two functions, so we must use the Chain Rule:

$$\frac{dy}{dx} = \sec^2\left(\frac{\sin x}{x}\right) \cdot \frac{d}{dx}\left(\frac{\sin x}{x}\right).$$

Now we have to use the Quotient Rule to find the derivative of the expression on the right.

Example 9

Solution

Using the Quotient Rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \sec^2\left(\frac{\sin x}{x}\right)\cdot\frac{\frac{d}{dx}\sin(x)\cdot x - \sin(x)\frac{d}{dx}(x)}{x^2}\\ &= \sec^2\left(\frac{\sin x}{x}\right)\cdot\frac{\cos(x)\cdot x - \sin(x)\cdot 1}{x^2}\\ &= \sec^2\left(\frac{\sin x}{x}\right)\left(\frac{x\cos(x) - \sin(x)}{x^2}\right).\end{aligned}$$

Example 10

Example

Find the d^2y/dx^2 if $y = x(2x+1)^4$.

Example 10

Solution

We first compute $\frac{dy}{dx}$ *using the Product Rule and the Chain Rule:*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x) \cdot (2x+1)^4 + x\frac{d}{dx}[(2x+1)^4] \\
&= (1) \cdot (2x+1)^4 + x \cdot 4(2x+1)^3 \cdot \frac{d}{dx}(2x+1) \\
&= (2x+1)^4 + x \cdot 4(2x+1)^3 \cdot (2) \\
&= (2x+1)^4 + 8x(2x+1)^3 \\
&= (2x+1)^3[(2x+1)+8x] \\
&= (2x+1)^3(10x+1).
\end{aligned}$$

Example 10

Solution

Now we compute the second derivative:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}[(2x+1)^3]\cdot(10x+1) + (2x+1)^3\cdot\frac{d}{dx}(10x+1)\\
&= \left[3(2x+1)^2\cdot\frac{d}{dx}(2x+1)\right]\cdot(10x+1) + (2x+1)^3\cdot(10)\\
&= \left[3(2x+1)^2\cdot(2)\right]\cdot(10x+1) + (2x+1)^3\cdot(10)\\
&= 6(2x+1)^2(10x+1) + 10(2x+1)^3\\
&= 2(2x+1)^2\left[3(10x+1) + 5(2x+1)\right]\\
&= 2(2x+1)^2\left(40x+8\right) = 16(2x+1)^2\left(5x+1\right).
\end{aligned}$$