Derivatives of Trigonometric Functions

William M. Faucette

University of West Georgia

Outline

1 Derivative of the Sine Function
2 Derivative of the Cosine Function
3 Derivatives of the Other Trigonometric Functions
4 Derivative of $\tan(x)$
5 Derivative of $\cot(x)$
6 Derivative of $\sec(x)$
7 Derivative of $\csc(x)$
8 Summary of Derivatives of Trigonometric Functions
9 Examples
10 Higher-Order Derivatives

Derivative of the Sine Function

Derivative of the Sine Function

We recall the formulas from precalculus for sine and cosine of a sum:

$$\begin{aligned}\sin(x+y) &= \sin(x)\cos(y)+\cos(x)\sin(y)\\ \cos(x+y) &= \cos(x)\cos(y)-\sin(x)\sin(y)\end{aligned}$$

We will use these to find the derivative of the sine function and the cosine function.

Derivative of the Sine Function

Let $f(x) = \sin x$. To compute the derivative of the sine function, we apply the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{[\sin(x)\cos(h)+\cos(x)\sin(h)]-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}
\end{aligned}$$

(Continued on the next slide)

Derivative of the Sine Function

Continuing the computation ...

$$\begin{aligned}
&\lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x) \lim_{h\to 0} \left[\frac{\cos(h)-1}{h}\right] + \cos(x) \lim_{h\to 0} \left[\frac{\sin(h)}{h}\right].
\end{aligned}$$

Fortunately, we computed these two limits in Section 2.3.

Derivative of the Sine Function

Finishing the computation with the values of the two limits computed earlier, we get

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \sin(x)\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] + \cos(x)\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right] \\ &= \sin(x)(0) + \cos(x)(1) \\ &= \cos(x).\end{aligned}$$

Derivative of the Cosine Function

Derivative of the Cosine Function

Let $f(x) = \cos x$. To compute the derivative of the sine function, we apply the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{[\cos(x)\cos(h)-\sin(x)\sin(h)]-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)-\sin(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h}
\end{aligned}$$

Derivative of the Cosine Function

Let $f(x) = \cos x$. To compute the derivative of the sine function, we apply the definition of the derivative:

$$
\begin{aligned}
\lim_{h \to 0} & \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h} \\
&= \lim_{h \to 0} \left\{ \cos(x) \left[\frac{cos(h) - 1}{h} \right] - \sin(x) \left[\frac{\sin(h)}{h} \right] \right\} \\
&= \cos(x) \lim_{h \to 0} \left[\frac{cos(h) - 1}{h} \right] - \sin(x) \lim_{h \to 0} \left[\frac{\sin(h)}{h} \right] \\
&= \cos(x)(0) - \sin(x)(1) \\
&= -\sin(x).
\end{aligned}
$$

Derivative of the Cosine Function

This gives us

Derivatives of $\sin(x)$ and $\cos(x)$

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Derivatives of the Other Trigonometric Functions

Derivatives of the Other Trigonometric Functions

To compute the derivatives of the remaining trigonometric functions, we use their definitions and the Quotient Rule.

Recall the definitions of the other four trigonometric functions:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

Derivative of $\tan(x)$

Derivative of $\tan(x)$

We compute the derivative of $\tan(x)$:

$$
\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}\sin(x)\cdot\cos(x) - \sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}
$$

Derivative of $\cot(x)$

Derivative of $\cot(x)$

We compute the derivative of $\cot(x)$:

$$\begin{aligned}
\frac{d}{dx}\cot(x) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \frac{\frac{d}{dx}\cos(x)\cdot\sin(x) - \cos(x)\cdot\frac{d}{dx}\sin(x)}{\sin^2(x)} \\
&= \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)} \\
&= \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} \\
&= -\csc^2(x).
\end{aligned}$$

Derivative of $\sec(x)$

Derivative of $\sec(x)$

We compute the derivative of $\sec(x)$:

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}(1)\cdot\cos(x) - (1)\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{(0)\cdot\cos(x) - (1)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} \\
&= \sec(x)\tan(x).
\end{aligned}$$

Derivative of $\csc(x)$

Derivative of $\csc(x)$

We compute the derivative of $\csc(x)$:

$$\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{\frac{d}{dx}(1)\cdot\sin(x) - (1)\frac{d}{dx}\sin(x)}{\sin^2(x)} \\
&= \frac{(0)\cdot\sin(x) - (1)(\cos(x))}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x).
\end{aligned}$$

Summary of Derivatives of Trigonometric Functions

Summary of Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Examples

Example 1

Example

Find dy/dx if

$$y = x^2 \cos x.$$

Example 1

Solution

We use the Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2)\cdot\cos x + x^2\cdot\frac{d}{dx}(\cos x)\\ &= 2x\cos x + x^2(-\sin x)\\ &= 2x\cos x - x^2\sin x.\end{aligned}$$

Example 2

Example

Find dy/dx if
$$y = \frac{\cos x}{1 + \sin x}.$$

Example 2

Solution

We use the Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)\cdot(1+\sin x) - \cos x\cdot\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} \\
&= \frac{-\sin x\cdot(1+\sin x) - \cos x\cdot\cos x}{(1+\sin x)^2} \\
&= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} \\
&= \frac{-\sin x - 1}{(1+\sin x)^2} = -\frac{1+\sin x}{(1+\sin x)^2} = -\frac{1}{1+\sin x}.
\end{aligned}$$

Example 3

Example

Find dy/dx if

$$y = x^2 \cos x - 2x \sin x - 2 \cos x.$$

Example 3

Solution

We compute:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(x^2\cos x - 2x\sin x - 2\cos x\right) \\
&= \frac{d}{dx}\left(x^2\cos x\right) - \frac{d}{dx}\left(2x\sin x\right) - \frac{d}{dx}\left(2\cos x\right) \\
&= \left[2x\cos x + x^2\left(-\sin x\right)\right] - \left[2\sin x + 2x\cos x\right] - 2(-\sin x) \\
&= -x^2\sin x.
\end{aligned}$$

Example 4

Example

Find dy/dx if

$$y = (\sec x + \tan x)(\sec x - \tan x).$$

Example 4

Solution

We compute

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\sec x + \tan x) \cdot (\sec x - \tan x) + \\
&\quad + (\sec x + \tan x) \cdot \frac{d}{dx}(\sec x - \tan x) \\
&= (\sec x \tan x + \sec^2 x) \cdot (\sec x - \tan x) + \\
&\quad + (\sec x + \tan x) \cdot (\sec x \tan x - \sec^2 x) \\
&= \sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \sec^2 x \tan x + \\
&\quad + \sec^2 x \tan x - \sec^3 x + \sec x \tan^2 x - \sec^2 x \tan x \\
&= 0.
\end{aligned}
$$

Example 4

Solution

We could have saved ourselves a lot of work by remembering that

$$(\sec x + \tan x)(\sec x - \tan x) = \sec^2 x - \tan^2 x = 1.$$

So the derivative is quite clearly zero.

Example 5

Example

Graph the curve

$$y = 1 + \cos x,$$

over the interval $-\frac{3\pi}{2} \leq x \leq 2\pi$, together with the tangent lines at $x = -\frac{\pi}{3}$ and $x = \frac{3\pi}{2}$.

Label the curve and tangent line with its equation.

Example 5

Solution

We first compute $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\sin x$$

$$\frac{dy}{dx}\left(-\frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{dy}{dx}\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{3\pi}{2}\right) = 1.$$

Example 5

Solution

At $x = -\frac{\pi}{3}$*, the slope is* $\frac{\sqrt{3}}{2}$ *and the point is* $\left(-\frac{\pi}{3}, \frac{3}{2}\right)$*. The equation of that tangent line is*

$$y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right).$$

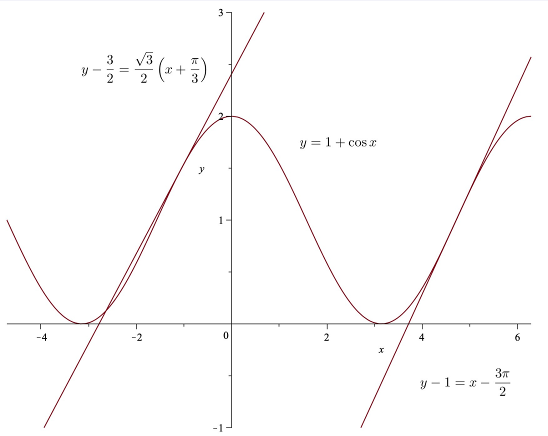
At $x = \frac{3\pi}{2}$*, the slope is* 1 *and the point is* $\left(\frac{3\pi}{2}, 1\right)$*. The equation of that tangent line is*

$$y - 1 = x - \frac{3\pi}{2}.$$

The sketch of the curve and the two tangent lines are on the next slide.

Example 5

Solution



Example 6

Example

Find all points on the curve $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

Example 6

Solution

The slope of the line $y = 2x$ is 2, so we have to find all the values of x in the interval $(-\pi/2, \pi/2)$ for $\frac{d}{dx}\tan x = \sec^2 x$ equals 2.

We solve

$$\begin{aligned}
\sec^2 x &= 2 \\
\sec x &= \sqrt{2} \\
\cos x &= \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \\
x &= \pm\frac{\pi}{4}
\end{aligned}$$

Example 6

Solution

For $x = -\pi/4$*, we get the point* $(-\pi/4, -1)$ *and the line is*

$$y + 1 = 2\left(x + \frac{\pi}{4}\right).$$

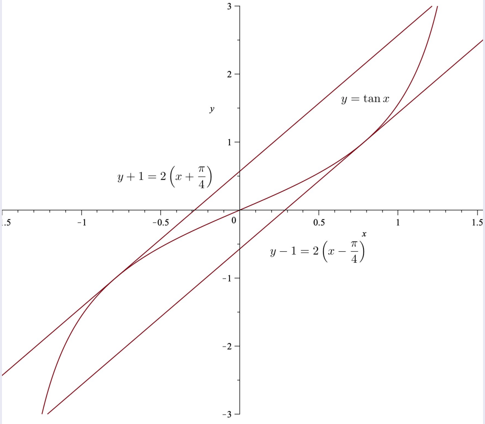
For $x = \pi/4$*, we get the point* $(\pi/4, 1)$ *and the line is*

$$y - 1 = 2\left(x - \frac{\pi}{4}\right).$$

The curve and the two tangent lines are graphed on the next slide.

Example 6

Solution



Higher-Order Derivatives

Example 6

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

See the table on the next slide.

Example 6

Higher Derivatives of $\sin x$ and $\cos x$

$$\begin{aligned} y &= \sin x & \qquad y &= \cos x \\ \frac{dy}{dx} &= \cos x & \qquad \frac{dy}{dx} &= -\sin x \\ \frac{d^2y}{dx^2} &= -\sin x & \qquad \frac{d^2y}{dx^2} &= -\cos x \\ \frac{d^3y}{dx^3} &= -\cos x & \qquad \frac{d^3y}{dx^3} &= \sin x \\ \frac{d^4y}{dx^4} &= \sin x & \qquad \frac{d^4y}{dx^4} &= \cos x \end{aligned}$$

Example 6

Example

For $y = \sin x$, find
$$\frac{d^{59}y}{dx^{59}}.$$

Example 6

Solution

Since the derivatives of $\sin x$ *repeat after a cycle of length 4, we know that*

$$\begin{aligned}\frac{d^{59}y}{dx^{59}} &= \frac{d^{4\cdot 14+3}y}{dx^{4\cdot 14+3}} \\ &= \frac{d^3y}{dx^3} \\ &= -\cos x.\end{aligned}$$