Defining the Derivative

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Outline

Tangent Lines

Tangent Lines

Definition

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h}$$

is a difference quotient with increment h.

Tangent Lines

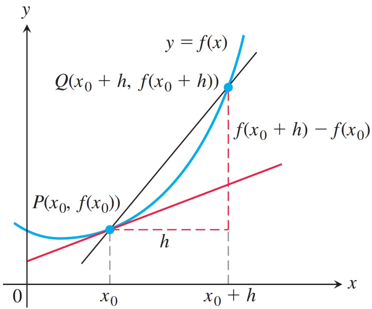

Figure: Slope of the tangent line at P

To find the slope of the tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$.

Tangent Lines

Definition

Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Tangent Lines

Definition

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{x - a}$$

provided this limit exists.

Example

Example 1

Example

Find an equation for the tangent line to the curve $y = 4 - x^2$ at the point $(-1, 3)$. Then sketch the curve and tangent line together.

Example 1

Solution

We compute the slope of the tangent line:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}\\
&= \lim_{h\to 0}\frac{[4-(-1+h)^2]-3}{h}\\
&= \lim_{h\to 0}\frac{[4-(1-2h+h^2)]-3}{h}\\
&= \lim_{h\to 0}\frac{2h-h^2}{h} = \lim_{h\to 0}\frac{h(2-h)}{h}\\
&= \lim_{h\to 0}(2-h) = 2.
\end{aligned}$$

Example 1

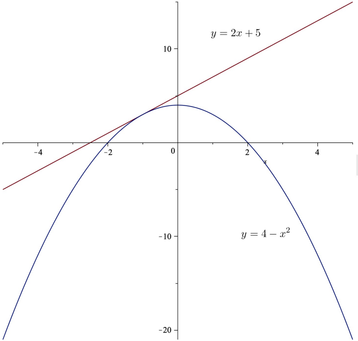

Figure: Sketch for Example 1

The equation of the tangent line is then

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - (-1))$$
$$y = 2x + 5.$$

Rates of Change: Derivative at a Point

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Recall the expression

$$\frac{f(a+h)-f(a)}{h}, \quad h \neq 0$$

is the **difference quotient of f at a with increment h.**

Rates of Change: Derivative at a Point

Definition

The **derivative of a function** f **at a point** a, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

Rates of Change: Derivative at a Point

The derivative has another interpretation in certain physical applications.

If we interpret the difference quotient as the average rate of change of the function f on the interval $[x_0, x_0 + h]$, then the derivative $f'(x_0)$ is the **instantaneous rate of change of f at x_0.**

Example

Example 2

Example

A rock falls $y = 16t^2$ feet during the first t seconds. What is the rock's exact speed after 1 second?

Example 2

Solution

With $f(t) = 16t^2$*, the difference quotient*

$$\frac{f(1+h) - f(1)}{h}$$

is the average velocity over the interval $[1, 1+h]$*.*

Example 2

Solution

If we then let $h \to 0$, we get the rock's exact speed:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{16(1+h)^2-16}{h} \\
&= \lim_{h\to 0} \frac{16(1+2h+h^2)-16}{h} \\
&= \lim_{h\to 0} \frac{16+32h+16h^2-16}{h} \\
&= \lim_{h\to 0} \frac{32h+16h^2}{h} = \lim_{h\to 0} \frac{h(32+16h)}{h} \\
&= \lim_{h\to 0} 32+16h = 32 \ ft/s.
\end{aligned}$$