A Preview of Calculus

William M. Faucette

University of West Georgia

Outline

The Tangent Problem and Differential Calculus

The Tangent Problem and Differential Calculus

Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

The Tangent Problem and Differential Calculus

On the graph $y = f(x)$, the average rate of change over the interval $[x_1, x_2]$ is the slope of the line between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. This is a **secant line** to the graph.

See the sketch on the next slide.

The Tangent Problem and Differential Calculus

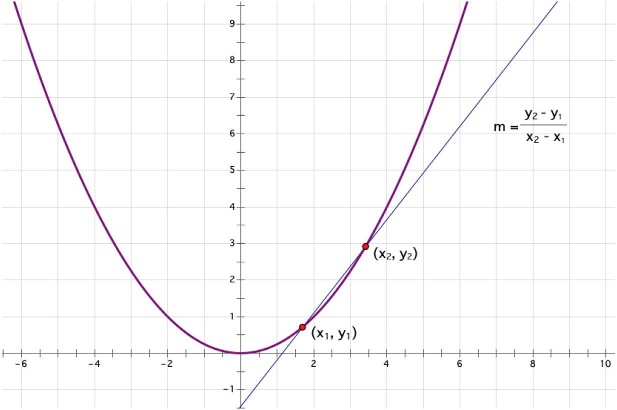

Figure: Sketch of a Secant Line

The Tangent Problem and Differential Calculus

The secant to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

This is the average rate of change of f on the interval between a and x.

The Tangent Problem and Differential Calculus

A tangent line to a curve meets the curve at only one point.

In order to find the slope of the tangent line, we look at the slopes of a secant lines from a fixed point P_0 on the curve to a variable point P on the curve.

Then we will let the point P get closer and closer to P_0 and observe what happens to the slopes.

The Tangent Problem and Differential Calculus

The slope of the tangent line to the graph at a measures the rate of change of the function at a.

This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$.

Differential calculus is the field of calculus concerned with the study of derivatives and their applications.

Example

Example 1

Example

Find the slope of the tangent line to the parabola

$$y = f(x) = x^2 - x$$

at the point $P_0(3, 6)$ by analyzing slopes of secant lines through $P_0(3, 6)$. Write an equation for the tangent line to the parabola at this point.

Example 1

Solution

We compute the slopes of secant lines through $P_0(3,6)$ *and* $P(3+h, f(3+h))$.

$$\begin{aligned}
\frac{f(3+h)-f(3)}{h} &= \frac{[(3+h)^2-(3+h)]-[3^2-3]}{h} \\
&= \frac{[9+6h+h^2-(3+h)]-[6]}{h} \\
&= \frac{6+5h+h^2-6}{h} \\
&= \frac{h(5+h)}{h} = 5+h.
\end{aligned}$$

Example 1

Solution

We form a table for values of h getting closer and closer to zero. This corresponds to the point P getting closer and closer to the point P_0.

h	Slope of Secant Line
1	6
0.1	5.1
0.01	5.01
0.001	5.001
0.0001	5.0001

Example 1

Solution

As h gets closer and closer to zero, P is getting closer and closer to the point P_0. The secant lines between P_0 and P get closer and closer to a tangent line at P_0. So, the slopes of the secant lines between P_0 and P get closer and closer to the slope of the tangent line at P_0.

Looking at the table, we see that the slopes of the secants lines are getting closer and closer to 5. So, the slope of the tangent line is 5.

Example 1

Solution

The equation of the tangent line to the curve $y = f(x) = x^2 - x$ *at the point* $(3, 6)$ *is*

$$\begin{aligned} y - 6 &= 5(x - 3) \\ y - 6 &= 5x - 15 \\ y &= 5x - 9. \end{aligned}$$

Rates of Change and Tangent Lines

Rates of Change and Tangent Lines

The average rate of change of f on the interval $[x, x+h]$ is defined to be

$$\frac{f(x+h) - f(x)}{h}.$$

As h gets closer and closer to zero, the average rate of change of the function f on the interval $[x, x+h]$ gets closer and closer to the actual rate of change of the function f at x. This is the **instantaneous rate of change** of f at x.

Rates of Change and Tangent Lines

The instantaneous rate of change of a function f at $x = a$ is the slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$.

First, we have to make precise what it means for the average values of a function to get "closer and closer" to something as h, the length of the interval, gets "closer and closer" to zero.

This involves the concept of a **limit**. We will take up this concept in the next section.

Average and Instantaneous Velocity

Average and Instantaneous Velocity

Suppose a moving object has traveled a distance $f(t)$ at time t. The average velocity during the time interval $[t_1, t_2]$ is the change in distance divided by the change in time. The unit of measure is length per unit time.

Average velocity

When $f(t)$ measures the distance traveled at time t, the average velocity over the interval $[t_1, t_2]$ is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

Example 1

Example 1

Example

A solid object dropped from rest near the surface of the earth and allowed to fall freely will fall according to the following equation:

$$s(t) = 16t^2$$

where t is measured in seconds and $s(t)$ is measured in feet. What is the object's average velocity during the first 2 sec of fall? During the time between $t = 1$ sec and $t = 2$ sec.

Example 1

Solution

The object's average velocity during the first 2 sec of fall is

$$\frac{s(2)-s(0)}{2-0} = \frac{16(2)^2-16(0)^2}{2-0} = \frac{64}{2} = 32 \ ft/s.$$

The object's average velocity during the time between $t = 1$ sec and $t = 2$ sec.

$$\begin{aligned}\frac{s(2)-s(1)}{2-1} &= \frac{16(2)^2-16(1)^2}{2-1} \\ &= \frac{64-16}{2-1} = 48 \ ft/s.\end{aligned}$$

Example 2

Example 2

Example

A solid object dropped from rest near the surface of the earth and allowed to fall freely will fall according to the following equation:

$$s(t) = 16t^2$$

where t is measured in seconds and $s(t)$ is measured in feet. Find the velocity of the falling rock in Example 1 between time $t = 1$ and $t = 1 + h$.

Example 2

Solution

The velocity of the falling rock in Example 1 between time $t = 1$ and $t = 1 + h$ is

$$\begin{aligned}
\frac{s(1+h) - s(1)}{(1+h) - 1} &= \frac{16(1+h)^2 - 16(1)^2}{(1+h) - 1} \\
&= \frac{16(1 + 2h + h^2) - 16}{h} \\
&= \frac{16 + 32h + 16h^2 - 16}{h} = \frac{32h + 16h^2}{h} \\
&= \frac{h(32 + 16h)}{h} = 32 + 16h.
\end{aligned}$$

Example 2

Solution

For values of h getting closer and closer to zero, we get

h	*Average velocity on* $[1, 1+h]$
1	*48*
0.1	*33.6*
0.01	*32.16*
0.001	*32.016*
0.0001	*32.0016*

The Area Problem and Integral Calculus

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve.

This leads us to ask the question: How can we find the area between the graph of a function and the x-axis over an interval.

See the sketch on the next slide.

The Area Problem and Integral Calculus

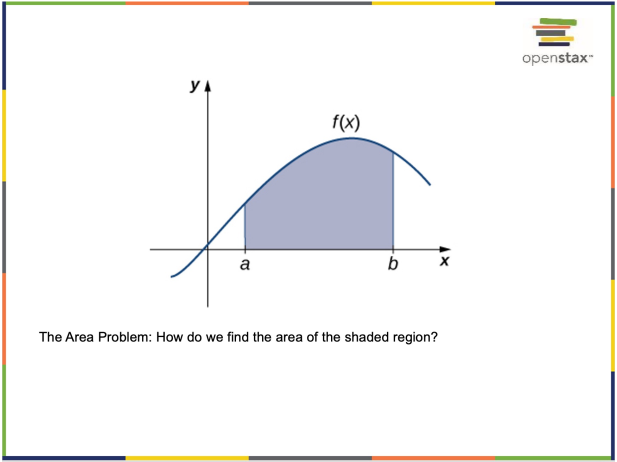

Figure: The Area Problem

As with the velocity question, we first try to approximate the solution.

We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles.

See the sketch on the next slide.

The Area Problem and Integral Calculus

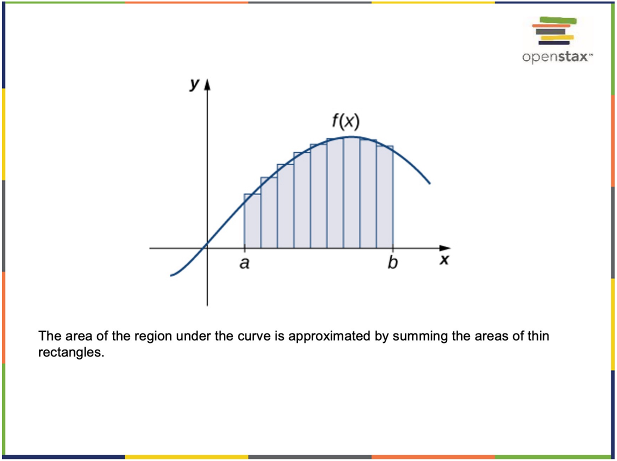

Figure: The Area Problem

The Area Problem and Integral Calculus

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the x-axis over the interval $[a, b]$.

Once again, we find ourselves taking a limit.

Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Other Aspects of Calculus

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions.

We might want to graph real-value functions of two variables or determine volumes of solids.

Other Aspects of Calculus

These are only a few of the types of questions that can be asked and answered using **multivariable calculus**.

Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables.

However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.