Derivatives and the Shape of a Graph

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Outline

First Derivative Test

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Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

- if f' changes from negative to positive at c, then f has a local minimum at c;
- if f' changes from positive to negative at c, then f has a local maximum at c;
- if f' does not change sign at c (that is, f is positive on both sides of c or negative on both sides), then f has no local extremum at c.

First Derivative Test

Proof.

If f' changes from negative to positive at c, then f is decreasing to the left of c and increasing to the right of c. This makes c a local minimum.

If f' changes from positive to negative at c, then f is increasing to the left of c and decreasing to the right of c. This makes c a local maximum.

If f' does not change sign across c, then f is increasing on both sides of c or f is decreasing on both sides of c. In either case, c is not an extremum. □

First Derivative Test

Problem-Solving Strategy: Using the First Derivative Test

Consider a function f that is continuous over an interval I.

1. Find all critical points of f and divide the interval I into smaller intervals using the critical points as endpoints.
2. Analyze the sign of f' in each of the subintervals. If f' is continuous over a given subinterval, then the sign of f' can be determined by choosing an arbitrary test point x in that subinterval and by evaluating the sign of f' at that test point. Use the sign analysis to determine whether f is increasing or decreasing over that interval.
3. Use First Derivative Test and the results of step 2 to determine whether f has a local maximum, a local minimum, or neither at each of the critical points.

Example

Example 2

Example

Find the interval(s) on which

$$f(x) = x^{1/3}(x+8)$$

is increasing and the interval(s) on which it is decreasing.

Example 2

Solution

First we compute the derivative.

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3}(x+8) + x^{1/3}(1) = \frac{1}{3}x^{-2/3}[(x+8)+3x] \\ &= \frac{1}{3}x^{-2/3}(4x+8) = \frac{4}{3}x^{-2/3}(x+2). \end{aligned}$$

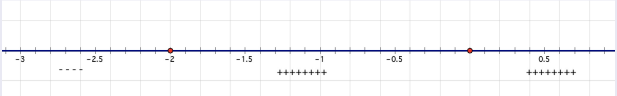
We see that f' is undefined at $x = 0$ and f' is zero at $x = -2$.

Example 2

Solution

We put the critical points of f on a number line and then evaluate f' at test points to determine the sign of f' on each interval.

Figure: *Sign of $f'(x)$*



Since $f' < 0$ on the interval $(-\infty, -2)$, the function $f(x)$ is decreasing on this interval. Since $f' > 0$ on the interval $(-2, 0)$, the function $f(x)$ is increasing on this interval. Since $f' > 0$ on the interval $(0, \infty)$, the function $f(x)$ is increasing on this interval.

Example 2

Solution

Since the derivative changes from being negative to being positive across $x = -2$, $x = -2$ *is a local minimum for* f.

Since the derivative does not change sign across $x = 0$, $x = 0$ *is not a local extremum for* f.

Example 3

Example

Find the intervals on which

$$f(x) = x^2 \ln x$$

is increasing and decreasing. Identify the function's local and absolute extreme values, if any, saying where they occur.

Example 3

Solution

First, the domain of f is all positive numbers.

Next we compute the derivative.

$$\begin{aligned} f'(x) &= 2x \ln x + x^2 \cdot \frac{1}{x} \\ &= 2x \ln x + x = x(2 \ln x + 1). \end{aligned}$$

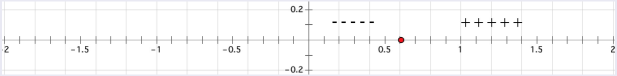
We see that f' is zero at $x = e^{-1/2}$.

Example 3

Solution

We put the critical points of f on a number line and then evaluate f' at test points to determine the sign of f' on each interval.

Figure: *Sign of $f'(x)$*



Since $f' < 0$ on the interval $(0, e^{-1/2})$, the function $f(x)$ is decreasing on this interval. Since $f' > 0$ on the interval $(e^{-1/2}, \infty)$, the function $f(x)$ is increasing on this interval.

Example 3

Solution

Since the derivative changes from being negative to being positive across $x = e^{-1/2}$*, the function* f *has a local minimum at* $x = e^{-1/2}$*. In fact, since the function is always decreasing and then always increasing,* $x = e^{-1/2}$ *is an absolute minimum. The absolute minimum value of this function is* $f(e^{-1/2}) = -\frac{1}{2e} \approx -0.184$*.*

Concavity and Curve Sketching

Concavity and Curve Sketching

Definition

Let f be a function that is differentiable over an open interval I.

- If f' is increasing over I, we say f is **concave up** over I.
- If f' is decreasing over I, we say f is **concave down** over I.

Test for Concavity

Test for Concavity

Since the first derivative tells if a function is increasing or decreasing, the first derivative of f' (that is, f'') tells you if f' is increasing or decreasing.

Test for Concavity

Let $y = f(x)$ be twice differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f is concave up over I.
2. If $f'' < 0$ on I, the graph of f is concave down over I.

Example

Example 1

Example

Find the interval(s) on which the function

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + \frac{1}{3}$$

is concave up and the interval(s) on which the function is concave down.

Example 1

Solution

We first compute the first and second derivatives.

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + \frac{1}{3}$$
$$f'(x) = x^2 - x - 2$$
$$f''(x) = 2x - 1.$$

We see that $f'' < 0$ *if* $x < \frac{1}{2}$ *and* $f'' > 0$ *if* $x > \frac{1}{2}$.

So, f *is concave down on the interval* $(-\infty, 1/2)$ *and* f *is concave up on the interval* $(1/2, \infty)$.

Inflection Points

Inflection Points

Definition

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an **inflection point** of f .

An inflection point is a point where the graph changes from being concave up to concave down or changes from being concave down to concave up.

At an inflection point, either f'' is undefined or f'' is zero.

Example

Example 2

Example

Find the coordinates of any local and absolute extreme points and inflection points of the function

$$y = x^4 - 2x^2.$$

Example 2

Solution

We first compute the first and second derivatives.

$$\begin{aligned} y &= x^4 - 2x^2 \\ y' &= 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1) \\ y'' &= 12x^2 - 4 = 4(3x^2 - 1). \end{aligned}$$

Setting y' equal to zero, we see the critical points are $x = 0, -1, 1$. Setting y'' equal to zero, we see the possible inflection points are $x = \pm\sqrt{3}/3$.

Example 2

Solution

Using the First Derivative Test with the derivative

$$y' = 4x(x-1)(x+1),$$

we find that

- f' *is negative for* $x < -1$;
- f' *is positive for* $-1 < x < 0$;
- f' *is negative for* $0 < x < 1$;
- f' *is positive for* $1 < x < \infty$.

The First Derivative Test tells us that f *has a local minimum at* $x = -1$, f *has a local maximum at* $x = 0$, *and* f *has a local minimum at* $x = 1$.

Example 2

Solution

We determine where the second derivative

$$y'' = 4(3x^2 - 1),$$

is zero or undefined to find the points where the second derivative can change sign. Then we choose test points in each interval to determine the sign of the second derivative on that interval.

Example 2

Solution

The second derivative is

$$y'' = 4(3x^2 - 1).$$

We find that

- f'' *is positive for* $x < -\frac{1}{\sqrt{3}}$
- f'' *is negative for* $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
- f'' *is positive for* $x > \frac{1}{\sqrt{3}}$.

From this, we see that the graph is concave up on the intervals $(-\infty, -\frac{1}{\sqrt{3}})$ *and* $(\frac{1}{\sqrt{3}}, \infty)$ *and the graph is concave down on the interval* $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

The Second Derivative Test

The Second Derivative Test

The Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous on an open interval that contains $x = c$.

1. If $f''(c) > 0$, then f has a local minimum at $x = c$.
2. If $f''(c) < 0$, then f has a local maximum at $x = c$.
3. If $f''(c) = 0$, then the test is inconclusive. The function f may have a local maximum, a local minimum, or neither.

Example

Example 3

We use the function from Example 2 as an example of the Second Derivative Test.

Example

Use the Second Derivative Test to find and classify the local extrema of the function

$$y = x^4 - 2x^2.$$

Example 3

Solution

From our preceding example, we found the critical points are $x = -1, 0, 1$.

The second derivative is

$$y'' = 12x^2 - 4.$$

We apply the Second Derivative Test to classify these critical points.

Example 3

Solution

We determine the sign of f'' at each of the critical points.

- $f''(-1) = 8 > 0$
- $f''(0) = -4 < 0$
- $f''(1) = 8 > 0.$

From this we see that $x = -1$ is a local minimum, $x = 0$ is a local maximum, and $x = 1$ is a local minimum, just as we found before.