Max-Min Worksheet 4

Problem. We need to design a cylindrical can with radius r and height h. The top and bottom must be made of copper, which will cost 2 cents per square inch. The curved side is to be made of aluminum, which will cost 1 cent per square inch. We seek the dimensions that will maximize the volume of the can. The only constraint is that the total cost of the can is to 300π cents.

Solution. First, we draw a picture:

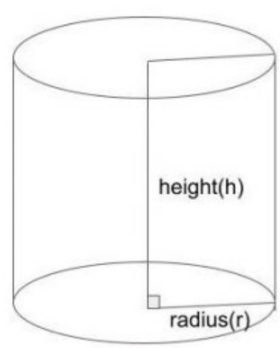

Figure 1: A Labeled Cylinder

Using the variables in Figure 1, we wish to maximize the volume of the can, which is $V = \pi r^2 h$. This function has too many variables, so we must find a relationship between r and h.

We see that the area of the top of the can is πr^2 square inches, the area of the bottom of the can is πr^2 square inches, and the area of the curved side of the can is $2\pi rh$ square inches.

So, we have the cost of the top is πr^2 square inches times 2 cents per square inch, which is $2\pi r^2$ cents. Similarly, the cost of the bottom is πr^2 square inches times 2 cents per square inch, which is $2\pi r^2$ cents. Finally, the cost of the side is $2\pi rh$ square inches times 1 cent per square inch, which is $2\pi rh$ cents. So, the total cost of the can is

$$C = 4\pi r^2 + 2\pi rh$$

cents. Since this must be 300π cents, we have

$$4\pi r^2 + 2\pi rh = 300\pi.$$

Solving this equation for h

$$\begin{aligned}
4\pi r^2 + 2\pi rh &= 300\pi \\
2r^2 + rh &= 150 \\
h &= \frac{150 - 2r^2}{r} = \frac{150}{r} - 2r.
\end{aligned}$$

Substituting this into the volume equation, we get

$$\begin{aligned}
V &= \pi r^2 h \\
&= \pi r^2 \left(\frac{150}{r} - 2r \right) \\
&= \pi r \left(150 - 2r^2 \right).
\end{aligned}$$

For our interval, we must have both $r \geq 0$ and $h \geq 0$. So, we must have

$$\frac{150}{r} - 2r \geq 0,$$

which means $r \leq 5\sqrt{3}$. So, we want to maximize the function on the interval $[0, 5\sqrt{3}]$.

Taking the derivative, we get

$$V' = \pi \left(150 - 2r^2 \right) + \pi r(-4r) = 150\pi - 6\pi r^2.$$

Setting this equal to zero and solving, we get the critical point $r = 5$ in the interval $[0, 5\sqrt{3}]$.

Taking the second derivative, we get

$$V'' = -12\pi r,$$

which is negative at our critical point. So, by the second derivative test, $r = 5$ is a local maximum. Since V is always concave down on the interval $[0, 5\sqrt{3}]$, this must actually be a global maximum on that interval.

So, in order to make the can have the largest possible volume, we must let the radius be 5 inches and the height be

$$h = \frac{150}{r} - 2r = \frac{150}{5} - 2(5) = 20 \text{ inches}.$$