Max-Min Worksheet 2

Problem. A piece of sheet metal is rectangular, 5 ft wide and 8 ft long. Congruent squares are to be cut from its four corners. The resulting piece of metal is to be folded and welded to form an open-topped box. How should this be done to get a box of largest possible volume?

Solution. First, we draw a picture:

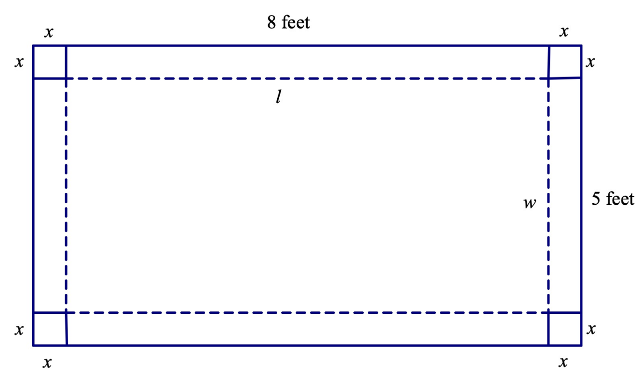

Figure 1: Rectangle with Four Corner Squares Removed

We wish to maximize the volume of this rectangular box, the volume of which is length times width times height.

Looking from left to right across the rectangle, we have

$$\ell + 2x = 8.$$

Looking from top to bottom across the rectangle, we have

$$w + 2x = 5.$$

Using the variables in Figure 1, the volume is

$$V = xw\ell = x(5 - 2x)(8 - 2x).$$

Each dimension must be nonnegative, so $x \geq 0$, $5 - 2x \geq 0$, and $8 - 2x \geq 0$. Hence, x must satisfy $0 \leq x \leq 5/2$.

We first multiply V out:

$$V = x(5 - 2x)(8 - 2x) = 4x^3 - 26x^2 + 40x$$

Taking the derivative of V, we get

$$\frac{dV}{dx} = 12x^2 - 52x + 40.$$

Setting dV/dx equal to zero and solving, we find the critical points $x = 1$ and $x = 10/3$, but this last critical point is outside the interval where x must be.

We know that this continuous function on this closed interval must have both a maximum value and a minimum value. By Fermat's Theorem, these values can only occur at the critical point(s) or the endpoints, 0 and 5/2. Since V is zero at both endpoints and $V(1) = 18$, this is the maximum value of V on the interval $[0, 5/2]$.

The problem how we should cut the piece of metal. The answer is: Cut squares of side length 1 ft from each corner. (This will give a box of volume 18 ft^3.)