The Definite Integral

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Outline

The Definite Integral

- As usual, you should read section 5.3 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

The Definition of the Definite Integral

Definition

Let $f(x)$ be a bounded function defined on a closed interval $[a, b]$. The **definite integral of f over $[a, b]$** is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1} f(c_k)\,\Delta x_k,$$

provided this limit exists. Here, $\|P\|$ is the **mesh** or **norm** of the partition. It is the largest of the Δx_k's.

The Definition of the Definite Integral

If the partition is regular so that each subrectangle has width $\Delta x = \frac{b-a}{n}$, then letting $\|P\|$ go to zero is the same as letting n go to infinity. This simply means you're dividing up the interval $[a, b]$ into more and more pieces.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x,$$

The Anatomy of a Definite Integral

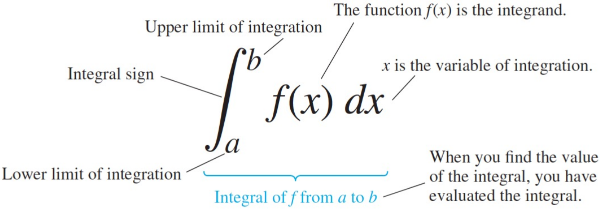


Figure: Parts of a Definite Integral

Definite Integral over a Regular Partition

Let f be a bounded function on the interval $[a, b]$.

If the partition is regular, then all the subrectangles have width $\Delta x = \frac{b-a}{n}$.

Then

$$x_k = a + k\,\Delta x = a + k\left(\frac{b-a}{n}\right).$$

The Riemann sum (using $c_k = x_k$) then becomes

$$\sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right).$$

Definite Integral over a Regular Partition

So, if the partition is regular, the definition of the definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right).$$

Do not be intimidated by this notation. All you're doing is taking the height of a rectangle, multiplying by its width to get its area, and then adding up all those areas. Finally, you're letting the number of rectangles go to infinity.

The Dummy Variable

The variable appearing in a definite integral is a **dummy variable**. The integrals

$$\int_a^b f(t)\,dt, \quad \int_a^b f(u)\,du, \quad \int_a^b f(x)\,dx$$

all mean the same thing.

Which Functions are Integrable?

Theorem

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

Which Functions are Integrable?

Define a function f on the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is not integrable on the interval $[0, 1]$.

(This function is discontinuous everywhere.)

Properties of Definite Integrals

Order of integration: $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Zero Width Interval: $$\int_a^a f(x)\,dx = 0$$

Constant Multiple: $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

Sum: $$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Difference: $$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Properties of Definite Integrals

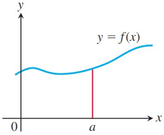
Additivity: $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Max-Min Inequality: If $m \leq f(x) \leq M$ on $[a, b]$, then
$$m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M \cdot (b-a)$$

Domination: If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$
If $f \geq g$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

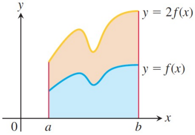
Properties of Definite Integrals

Figure: Sketch of Properties



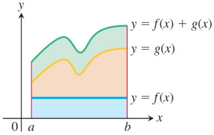
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



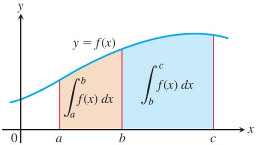
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



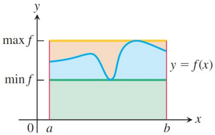
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



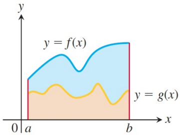
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$$



(f) *Domination:*

If $f(x) \ge g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Properties of Definite Integrals

Definition

If $f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve** $y = f(x)$ **over** $[a, b]$ is

$$A = \int_a^b f(x)\,dx.$$

Example 1

Example

Let $m > 0$. Compute $\int_0^b mx\,dx$ and find the area under $y = mx$ over the interval $[0, b]$, $b > 0$.

Solution

We compute the integral using a regular partition. We have $\Delta x = \frac{b-0}{n} = \frac{b}{n}$. *We have* $x_k = a + k\,\Delta x = k\frac{b}{n}$.

Using the right endpoint as c_k, *the Riemann sum is then*

$$\sum_{k=1}^{n} f\left(k\frac{b}{n}\right)\cdot\frac{b}{n} = \sum_{k=1}^{n} m\left(k\frac{b}{n}\right)\cdot\frac{b}{n} = \sum_{k=1}^{n}\left(\frac{mb^2}{n^2}\right)k.$$

Example 1

Solution

Now, we need to put this sum into what is called **closed form**. *That is, we have to add it up.*

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{mb^2}{n^2}\right)k &= \left(\frac{mb^2}{n^2}\right)\sum_{k=1}^{n}k \\ &= \left(\frac{mb^2}{n^2}\right)\frac{n(n+1)}{2} \\ &= \left(\frac{mb^2}{2}\right)\frac{n+1}{n}.\end{aligned}$$

Example 1

Solution

Finally, the integral is then the limit of the Riemann sums:

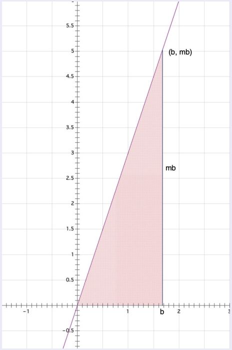
$$\begin{aligned}\int_0^b mx\,dx &= \lim_{n\to\infty}\left(\frac{mb^2}{2}\right)\frac{n+1}{n}\\ &= \frac{mb^2}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\\ &= \frac{mb^2}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\\ &= \frac{mb^2}{2}(1+0)\\ &= \frac{mb^2}{2}.\end{aligned}$$

Example 1

Solution

Of course, this is the correct answer. What we have just found is the area of this triangle.

Figure: *Sketch of* $y = mx$ *on* $[0, b]$



Example 2

Example

Let $m > 0$. Compute $\int_a^b mx\,dx$ and find the area under $y = mx$ over the interval $[a, b]$, $b > a > 0$.

Solution

We compute the integral using a regular partition. We have $\Delta x = \dfrac{b-a}{n}$.

We have $x_k = a + k\Delta x = a + k\dfrac{b-a}{n}$.

Using the right endpoint as c_k, *the Riemann sum is then*

$$\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\cdot\frac{b-a}{n} = \sum_{k=1}^{n} m\left(a + k\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$$
$$= \sum_{k=1}^{n} \frac{ma(b-a)}{n} + \left(\frac{m(b-a)^2}{n^2}\right)k$$

Example 2

Solution

We massage this computation a wee bit more:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{ma(b-a)}{n} + \left(\frac{m(b-a)^2}{n^2}\right) k \\
&= \sum_{k=1}^{n} \frac{ma(b-a)}{n} + \sum_{k=1}^{n} \left(\frac{m(b-a)^2}{n^2}\right) k \\
&= \frac{ma(b-a)}{n} \sum_{k=1}^{n} 1 + \left(\frac{m(b-a)^2}{n^2}\right) \sum_{k=1}^{n} k \\
&= \frac{ma(b-a)}{n} \cdot n + \left(\frac{m(b-a)^2}{n^2}\right) \frac{n(n+1)}{2} \\
&= ma(b-a) + \left(\frac{m(b-a)^2}{2}\right) \left(\frac{n+1}{n}\right)
\end{aligned}$$

Example 2

Solution

Now we can compute the integral by taking the limit as n goes to infinity:

$$\begin{aligned}\int_a^b mx\,dx &= \lim_{n\to\infty}\left[ma(b-a)+\left(\frac{m(b-a)^2}{2}\right)\left(\frac{n+1}{n}\right)\right]\\ &= ma(b-a)+\left(\frac{m(b-a)^2}{2}\right)\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\\ &= ma(b-a)+\frac{m(b-a)^2}{2}\\ &= \frac{1}{2}m(b-a)[2a+(b-a)]\\ &= m(b-a)\left(\frac{a+b}{2}\right).\end{aligned}$$

Example 2

Solution

Of course, this is the correct answer. What we have just found is the area of this trapezoid.

Figure: *Sketch of $y = mx$ on $[a, b]$*