

Sigma Notation and Limits of Finite Sums

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Sigma Notation and Limits of Finite Sums

- As usual, you should read section 5.2 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Finite Sums and Sigma Notation

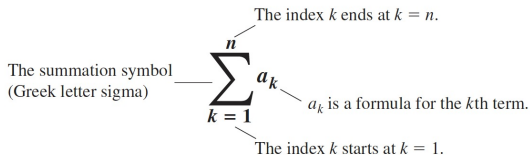
If we are going to be adding up more and more terms, we need some notation to deal with this. The notation is **sigma notation** and it looks like this:

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The sum on the right is the sum represented by the sigma notation on the left.

The letter k is the **index** of the sum.

Figure: Anatomy of a Summation



Examples of Sigma Notation

$$\sum_{k=1}^8 k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$

$$\sum_{k=1}^4 \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$\sum_{k=1}^5 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 62$$

Algebra Rules for Finite Sums

1. Sum Rule:
$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

2. Difference Rule:
$$\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$$

3. Constant Multiple Rule:
$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k$$

4. Constant Value Rule:
$$\sum_{k=1}^n c = n \cdot c$$

An Important Formula

Example

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Proof.

Let

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

Now reverse the order of the summands in S :

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$



An Important Formula

Proof.

We have

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n,$$

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Now, add the two sums from top to bottom

$$\begin{aligned} 2S &= \overbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}^{n \text{ times}} \\ &= n(n+1). \end{aligned}$$

Dividing the equation by 2 gives the result. □

More Important Formulas

The first n squares:
$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes:
$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Limits of Finite Sums

Start with the function $f(x) = x^2$ on the interval $[0, 1]$.

We divide the interval into n pieces of equal width. These points are

$$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1.$$

The set of these points is called a **partition** of the interval $[0, 1]$.

For notation, $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, etc. In general, we have $x_k = \frac{k}{n}$.

Limits of Finite Sums

We now need to choose a point in the k th interval where we will evaluate the function to get the height of the rectangle.

It doesn't matter which point you pick, so we'll pick the right endpoint, $\frac{k}{n}$. The height of the k th rectangle is then $f(\frac{k}{n})$.

The area of the k th rectangle is then $f(\frac{k}{n}) \cdot \frac{1}{n} = (\frac{k}{n})^2 \cdot \frac{1}{n} = \frac{k^2}{n^3}$.

The sum of the areas of these n rectangles is then

$$\sum_{k=1}^n \frac{k^2}{n^3}.$$

Limits of Finite Sums

We'll use our list of important formulas and the properties of sums to write this sum in **closed form**.

$$\begin{aligned}\sum_{k=1}^n \frac{k^2}{n^3} &= \frac{1}{n^3} \sum_{k=1}^n k^2 \\ &= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6n^2}.\end{aligned}$$

Limits of Finite Sums

Since the estimate of the area should get better as we divide the interval into more and more pieces, we take the limit as n goes to infinity.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} \cdot \frac{1/n^2}{1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{2 + 3\frac{1}{n} + \frac{1}{n^2}}{6} \\ &= \frac{2 + 3(0) + (0)}{6} \\ &= \frac{1}{3}.\end{aligned}$$

This is the area under the graph of $y = x^2$ over the interval $[0, 1]$.

Riemann Sums

Start with a bounded function $f(x)$ defined on an interval $[a, b]$.

We choose a **partition** $P = \{x_1, x_2, \dots, x_{n-1}\}$ between a and b so that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

The partition P divides the interval $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

The k th subinterval is $[x_{k-1}, x_k]$. We denote its width by $\Delta x_k = x_k - x_{k-1}$.

The largest value of Δx_k is the **mesh** or **norm** of the partition. It is denoted $\|P\|$.

Riemann Sums

In the interval $[x_{k-1}, x_k]$, we choose a point c_k . This is the point where we will evaluate f to get the height of the k th rectangle. So, the height of the k th rectangle is $f(c_k)$.

The area of the k th rectangle is $f(c_k) \Delta x_k$, and the sum of the areas of the n rectangles is

$$S_P = \sum_{k=1}^n f(c_k) \Delta x_k,$$

which is the **Riemann sum for f on the interval $[a, b]$ with respect to the partition P .**

This man's name is pronounced "Rē'· mǒn".

Riemann Sums

If all the subintervals have the same width, the partition is called **regular**. In this case, we denote the width of each rectangle is given by

$$\Delta x = \frac{b - a}{n}.$$

Also, in the case of a regular partition, we have

$$x_k = a + k \Delta x = a + k \frac{b - a}{n}$$