Indeterminate Forms and L'Hôpital's Rule Exponential Forms

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Outline

Indeterminate Forms and L'Hôpital's Rule

- As usual, you should read section 4.5 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Indeterminate Forms

There are (at least) seven **indeterminate forms**. These are limits that look like

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

When you have a limit of one of these forms, you have to put forth more effort to compute the limit.

This slide show will cover the last three indeterminate forms—the exponential indeterminate forms.

Exponential Indeterminate Forms

The remaining three indeterminate forms are all exponentials. These are all dealt with in the same way. Taking the natural logarithm of the expression yields and expression of the form $\infty \cdot 0$, which you compute as we did before. When you're finished, you have to remember that you took the natural logarithm to get this answer, so you must take the result and exponentiate it to get the original limit.

L'Hôpital's Rule: 0^0

To deal with limits of the form 0^0, you take the natural logarithm of the limit. This converts the limit to one of the form $0 \cdot \infty$. Compute this limit as before, then exponentiate to get the value of the original limit.

Example 7

Example

Compute

$$\lim_{x \to 0^+} x^x$$

Solution

We notice that this limit looks like 0^0*, which is an exponential indeterminate form. We set* $L = \lim_{x \to 0^+} x^x$*. Now, we take the natural logarithm of this equation.*

$$\begin{aligned} \ln(L) &= \ln(\lim_{x \to 0^+} x^x) \\ &= \lim_{x \to 0^+} \ln(x^x) \\ &= \lim_{x \to 0^+} x \ln(x). \end{aligned}$$

Example 7

Solution

This limit has the indeterminate form $0 \cdot (-)\infty$, *which we compute using the process presented before.*

$$\begin{aligned} \ln(L) &= \lim_{x \to 0^+} x \ln(x) \\ &= 0. \\ L &= e^0 = 1. \end{aligned}$$

So, the original limit is 1.

L'Hôpital's Rule: 1^∞

To deal with limits of the form 1^∞, you take the natural logarithm of the limit. This converts the limit to one of the form $\infty \cdot 0$. Compute this limit as before, then exponentiate to get the value of the original limit.

Example 8

Example

Compute

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Solution

We note that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1$*, so this limit looks like* 1^∞*, which is an exponential indeterminate form. We set* $L=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$*. Now, we take the natural logarithm of this equation.*

$$\begin{aligned}\ln(L) &= \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)\\ &= \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x\\ &= \lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right).\end{aligned}$$

Example 8

Solution

This limit has the form $\infty \cdot 0$, *which we compute using the process presented before.*
So, we have

$$\ln(L) = \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)$$
$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

Now, this limit looks like $0/0$ *and we can apply L'Hôpital's Rule directly.*

Example 8

Solution

We apply L'Hôpital's Rule for the indeterminate form $0/0$*:*

$$\begin{aligned}\ln(L) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \\ &= \lim_{x\to\infty} \frac{\left[\ln\left(1+\frac{1}{x}\right)\right]'}{(1/x)'} \\ &= \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}(1/x)'}{(1/x)'} \\ &= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} = 1.\end{aligned}$$

So, $L = e^1 = e$*. The original limit is* e*.*

L'Hôpital's Rule:∞^0

To deal with limits of the form ∞^0, you take the natural logarithm of the limit. This converts the limit to one of the form $0 \cdot \infty$. Compute this limit as before, then exponentiate to get the value of the original limit.

Example 9

Example

Compute

$$\lim_{x\to\infty} x^{1/x}.$$

Solution

We note that $\lim_{x\to\infty}\frac{1}{x}=0$*, so this limit looks like* ∞^0*, which is an exponential indeterminate form. We set* $L=\lim\limits_{x\to\infty} x^{1/x}$*. Now, we take the natural logarithm of this equation.*

$$\begin{aligned}\ln(L) &= \ln\left(\lim_{x\to\infty} x^{1/x}\right)\\ &= \lim_{x\to\infty}\ln\left(x^{1/x}\right)\\ &= \lim_{x\to\infty}\frac{1}{x}\ln(x)\\ &= \lim_{x\to\infty}\frac{\ln(x)}{x}.\end{aligned}$$

Example 9

Solution

We have already computed this limit in a preceding example, where we found that $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$.

So, we have

$$L = e^0 = 1$$

as our original limit.