Concavity and Curve Sketching

William M. Faucette

University of West Georgia

Outline

Concavity and Curve Sketching

- As usual, you should read section 4.4 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Concavity and Curve Sketching

Definition

The graph of a differentiable function $y = f(x)$ is

- **concave up** on an open interval I if f' is increasing on I;
- **concave down** on an open interval I if f' is decreasing on I;

The Second Derivative Test for Concavity

Since the first derivative tells if a function is increasing or decreasing, the first derivative of f' (that is, f'') tells you if f' is increasing or decreasing.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1

Example

Find the interval(s) on which the function

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + \frac{1}{3}$$

is concave up and the interval(s) on which the function is concave down.

Solution

We compute the second derivative.

$$f'(x) = x^2 - x - 2$$
$$f''(x) = 2x - 1.$$

We see that $f'' < 0$ *if* $x < \frac{1}{2}$ *and* $f'' > 0$ *if* $x > \frac{1}{2}$.

So, f *is concave down on the interval* $(-\infty, 1/2)$ *and* f *is concave up on the interval* $(1/2, \infty)$.

Points of Inflection

Definition

A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

A point of inflection is a point where the graph changes from being concave up to concave down or changes from being concave down to concave up.

At a point of inflection, either f'' is undefined or f'' is zero.

Example 2

Example

Find the coordinates of any local and absolute extreme points and inflection points of the function

$$y = x^4 - 2x^2.$$

Solution

We first compute the first and second derivatives.

$$\begin{aligned} y' &= 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1) \\ y'' &= 12x^2 - 4 = 4(3x^2 - 1). \end{aligned}$$

Setting y' equal to zero, we see the critical points are $x = 0, -1, 1$. Setting y'' equal to zero, we see the possible inflection points are $x = \pm\sqrt{3}/3$.

Example 2

Solution

Using the First Derivative Test, we find that

- f' *is negative for* $x < -1$;
- f' *is positive for* $-1 < x < 0$;
- f' *is negative for* $0 < x < 1$;
- f' *is positive for* $1 < x < \infty$.

The First Derivative Test tells us that f *has a local minimum at* $x = -1$, f *has a local maximum at* $x = 0$, *and* f *has a local minimum at* $x = 1$.

Example 2

Solution

We put the possible inflection points on a number line and choose test points to determine the sign of the second derivative on each interval. We find that

- f'' *is positive for* $x < -\frac{1}{\sqrt{3}}$
- f'' *is negative for* $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
- f'' *is positive for* $x > \frac{1}{\sqrt{3}}$.

From this, we see that the graph is concave up on the intervals $(-\infty, -\frac{1}{\sqrt{3}})$ *and* $(\frac{1}{\sqrt{3}}, \infty)$ *and the graph is concave down on the interval* $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Second Derivative Test for Local Extrema

Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Example 3

We use our previous example as an example of the Second Derivative Test.

Example

Use the Second Derivative Test to find and classify the local extrema of the function

$$y = x^4 - 2x^2.$$

Solution

From our preceding example, we found the critical points are $x = -1, 0, 1$. *We apply the Second Derivative Test to classify these critical points.*

- $f''(-1) = 8 > 0$
- $f''(0) = -4 < 0$
- $f''(1) = 8 > 0.$

From this we see that $x = -1$ *is a local minimum,* $x = 0$ *is a local maximum, and* $x = 1$ *is a local minimum, just as we found before.*

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Identify any asymptotes that may exist.
3. Find the derivatives y' and y''.
4. Find the critical points of f, if any, and identify the function's behavior at each one.
5. Find where the curve is increasing and where it is decreasing.
6. Find the points of inflection, if any occur, and determine the concavity of the curve.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example 4

Example

Graph the rational function

$$y = \frac{2x^2 + x - 1}{x^2 - 1}$$

using all the steps in the graphing procedure on the preceding slide.

Solution

We first do the precalculus. We simplify the function to get

$$\frac{2x^2+x-1}{x^2-1} = \frac{(2x-1)(x+1)}{(x-1)(x+1)} = \frac{2x-1}{x-1}.$$

Setting $x = 0$, we find a y-intercept at $y = 1$.

Setting the numerator equal to zero, we find x-intercepts at $x = \frac{1}{2}$.

Setting the denominator equal to zero, we find vertical asymptote at $x = 1$.

Taking the limit as $x \to \infty$, we find horizontal asymptote at $y = 2$.

Since we canceled the factor $x + 1$, there is a hole in the graph at $x = -1$. (The function has a removable discontinuity at $x = -1$.)

If you don't understand this, go to the Mathematics Tutoring Center.

Solution

Taking the derivative, we get

$$
\begin{aligned}
y' &= \frac{(2x^2+x-1)'(x^2-1)-(2x^2+x-1)(x^2-1)'}{(x^2-1)^2} \\
&= \frac{(4x+1)(x^2-1)-(2x^2+x-1)(2x)}{(x^2-1)^2} \\
&= \frac{(4x^3+x^2-4x-1)-(4x^3+2x^2-2x)}{(x^2-1)^2} \\
&= \frac{-x^2-2x-1}{[(x-1)(x+1)]^2} = \frac{-(x+1)^2}{(x-1)^2(x+1)^2} \\
&= -\frac{1}{(x-1)^2}.
\end{aligned}
$$

Solution

Taking the second derivative, we get $y'' = \frac{2}{(x-1)^3}$.
This derivative is never zero and is undefined at $x = 1$.
This derivative is negative for $x < 1$ *and positive for* $x > 1$.
So, $f'' < 0$ *on the interval* $(-\infty, 1)$ *and* $f'' > 0$ *on the interval* $(1, \infty)$.
So, f *is concave down on the interval* $(-\infty, 1)$ *and concave up on the interval* $(1, \infty)$.

Solution

Sketching the graph from the information we've gotten, we get

Figure: *Sketch of* $y = f(x)$