Monotonic Functions and the First Derivative Test

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Outline

The First Derivative Test

- As usual, you should read section 4.3 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Increasing Functions and Decreasing Functions

Another important consequence of the Mean Value Theorem is the following:

Corollary

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

- *If $f'(x) > 0$ at each point $x \in (a, b)$, the f is increasing on $[a, b]$.*
- *If $f'(x) < 0$ at each point $x \in (a, b)$, the f is decreasing on $[a, b]$.*

The sign of the first derivative tells you if the function is increasing or decreasing.

Example 1

Example

Find the interval(s) on which

$$g(x) = x^4 - 4x^3 + 4x^2$$

is increasing and the interval(s) on which it is decreasing.

Solution

First we compute the derivative.

$$\begin{aligned} g'(x) &= 4x^3 - 12x^2 + 8x \\ &= 4x(x-1)(x-2). \end{aligned}$$

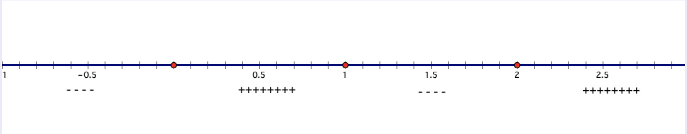
We see that g' exists everywhere and is zero at $x = 0, 1, 2$.

Example 1

Solution

We put the critical points of g on a number line and then evaluate g' at test points to determine the sign of g' on each interval.

Figure: *Sign of $g'(x)$*



Since $g' < 0$ on the interval $(-\infty, 0)$, the function $g(x)$ is decreasing on this interval. Since $g' > 0$ on the interval $(0, 1)$, the function $g(x)$ is increasing on this interval. Since $g' < 0$ on the interval $(1, 2)$, the function $g(x)$ is decreasing on this interval. Since $g' > 0$ on the interval $(2, \infty)$, the function $g(x)$ is increasing on this interval.

First Derivative Test for Local Extrema

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

- if f' changes from negative to positive at c, then f has a local minimum at c;
- if f' changes from positive to negative at c, then f has a local maximum at c;
- if f' does not change sign at c (that is, f is positive on both sides of c or negative on both sides), then f has no local extremum at c.

First Derivative Test for Local Extrema

Proof.

If f' changes from negative to positive at c, then f is decreasing to the left of c and increasing to the right of c. This makes c a local minimum.

If f' changes from positive to negative at c, then f is increasing to the left of c and decreasing to the right of c. This makes c a local maximum.

If f' does not change sign across c, then f is increasing on both sides of c or f is decreasing on both sides of c. In either case, c is not an extremum. □

Example 2

Example

Find the interval(s) on which

$$f(x) = x^{1/3}(x + 8)$$

is increasing and the interval(s) on which it is decreasing.

Solution

First we compute the derivative.

$$f'(x) = \frac{1}{3}x^{-2/3}(x + 8) + x^{1/3}(1) = \frac{1}{3}x^{-2/3}[(x + 8) + 3x]$$
$$= \frac{1}{3}x^{-2/3}(4x + 8) = \frac{4}{3}x^{-2/3}(x + 2).$$

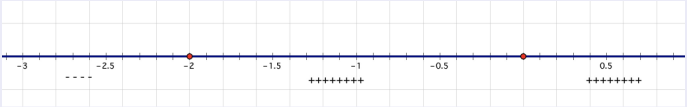
We see that f' is undefined at $x = 0$ and f' is zero at $x = -2$

Example 2

Solution

We put the critical points of f on a number line and then evaluate f' at test points to determine the sign of f' on each interval.

Figure: *Sign of $f'(x)$*



Since $f' < 0$ on the interval $(-\infty, -2)$, the function $f(x)$ is decreasing on this interval. Since $f' > 0$ on the interval $(-2, 0)$, the function $f(x)$ is increasing on this interval. Since $f' > 0$ on the interval $(0, \infty)$, the function $f(x)$ is increasing on this interval.

Example 2

Solution

Since the derivative changes from being negative to being positive across $x = -2$, $x = -2$ *is a local minimum for* f.

Since the derivative does not change sign across $x = 0$, $x = 0$ *is not a local extremum for* f.

Example 3

Example

Find the intervals on which

$$f(x) = x^2 \ln x$$

is increasing and decreasing. Identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

First, the domain of f is all positive numbers.
Next we compute the derivative.

$$\begin{aligned} f'(x) &= 2x \ln x + x^2 \cdot \frac{1}{x} \\ &= 2x \ln x + x = x(2 \ln x + 1). \end{aligned}$$

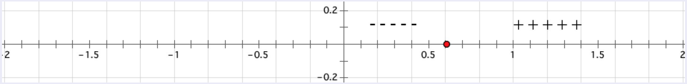
We see that f' is zero at $x = e^{-1/2}$.

Example 3

Solution

We put the critical points of f on a number line and then evaluate f' at test points to determine the sign of f' on each interval.

Figure: *Sign of $f'(x)$*



Since $f' < 0$ on the interval $(0, e^{-1/2})$, the function $f(x)$ is decreasing on this interval. Since $f' > 0$ on the interval $(e^{-1/2}, \infty)$, the function $f(x)$ is increasing on this interval.

Example 3

Solution

Since the derivative changes from being negative to being positive across $x = e^{-1/2}$, the function f has a local minimum at $x = e^{-1/2}$. In fact, since the function is always decreasing and then always increasing, $x = e^{-1/2}$ is an absolute minimum. The absolute minimum value of this function is $f(e^{-1/2}) = -\frac{1}{2e} \approx -0.184$.