The Mean Value Theorem

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Outline

The Mean Value Theorem

- As usual, you should read section 4.2 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Rolle's Theorem

Rolle's Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

What this says is that if a graph starts at a certain height and ends at that same height and is "nice" in between, then the derivative must be zero somewhere in between. That is, if the function goes up, then it must come back down, so it must turn around, i.e. the derivative is zero. If the function goes down, then it must come back up, so it must turn around, i.e. the derivative is zero.

Example 1

Example

Consider the function

$$f(x) = x^3 - 33x^2 + 216x = x(x-9)(x-24).$$

Show that this function is zero at $x = 0$, $x = 9$, and $x = 24$. Find the values of c guaranteed by Rolle's Theorem on the intervals $[0, 9]$ and $[9, 24]$.

Example 1

Solution

It's easy to see from the factored polynomial that it is zero at $x = 0$, $x = 9$, *and* $x = 24$. *So,* $f(0) = f(9) = 0$ *and* $f(9) = f(24) = 0$.

We take the derivative

$$y' = 3x^2 - 66x + 216 = 3(x - 4)(x - 18).$$

So, we see the derivative is zero at $x = 4$ *and* $x = 18$.

So, on the interval $[0, 9]$, *we have* $c = 4$ *and on the interval* $[9, 24]$ *we have* $c = 18$.

The Mean Value Theorem

The Mean Value Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there is at least one number c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The fraction on the left is the average rate of change of f on the interval $[a, b]$. The value on the right is the instantaneous rate of change of f at $x = c$. So, what this says is that if a function has an average rate of change over and interval $[a, b]$, then at some point inside the open interval (a, b), the function must actually be changing at that rate.

If you average sixty miles per hour on a trip, at some point during the trip you must be going sixty miles per hour.

Example 2

Example

Consider the function

$$f(x) = x^3 - x^2$$

on the interval $[-1, 2]$. Find the values of c guaranteed by the Mean Value Theorem.

Solution

This function is a polynomial, so it's continuous and differentiable everywhere. So, the function satisfies the hypotheses of the Mean Value Theorem.

We take the derivative: $f'(x) = 3x^2 - 2x$*. Then we compute the average rate of change of* f *on the interval:*

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - (-2)}{3} = \frac{6}{3} = 2.$$

So, we have to solve $f'(c) = 2$*.*

$$\begin{aligned} 3c^2 - 2c &= 2 \\ 3c^2 - 2c - 2 &= 0 \\ c &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)} = \frac{1}{3} \pm \frac{\sqrt{7}}{3}. \end{aligned}$$

Both these numbers lie in the interval $[-1, 2]$*, so we have* $c = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$*.*

Important Corollary

The importance of the Mean Value Theorem is in the use of the fact that c exists in proving other results, not in actually computing c.

For example, we already know that if a function is constant, then its derivative is zero.

One of the important results of the Mean Value Theorem is the converse of this statement:

Corollary

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

So, if the derivative of a function is zero in an interval, then the function is constant in that interval.

Another Important Corollary

Another important result of the Mean Value Theorem which follows immediately from the last one is this:

Corollary

If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is constant on (a, b).

So, if the two functions have the same derivative then those functions differ by a constant.