Extreme Values of Functions on Closed Intervals

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Outline

Extreme Values of Functions on Closed Intervals

- As usual, you should read section 4.1 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Absolute Maxima and Absolute Minima

Definition

Let f be a function with domain D. Then f has an **absolute maximum** or **global maximum** at a point c if $f(x) \leq f(c)$ for all x in D. That is, f has its largest value on D at the point c.

Let f be a function with domain D. Then f has an **absolute minimum** or **global minimum** at a point c if $f(x) \geq f(c)$ for all x in D. That is, f has its smallest value on D at the point c.

Examples

Example

1. The function $y = x^3$ on the domain $(-\infty, \infty)$ has no absolute extrema.
2. The function $y = x^3$ on the domain $[0, 2]$ has an absolute maximum of 8 at $x = 2$ and an absolute minimum of 0 at $x = 0$.
3. The function $y = x^3$ on the domain $(0, 2]$ has an absolute maximum of 8 at $x = 2$ and no absolute minimum.
4. The function $y = x^3$ on the domain $[0, 2)$ an absolute minimum of 0 at $x = 0$ and no absolute maximum.

The Extreme Value Theorem

We have the following very important theorem that tells us that maximum values and minimum values exist in certain circumstances.

Theorem (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$ and $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Proof.

A proof of this is beyond the scope of this course. □

If f is not continuous or we're not looking at a closed interval, there still may be extreme values, but their existence isn't guaranteed.

Local (Relative) Extreme Values

Definition

A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Together these are called **local** (or **relative) extrema**.

Finding Extrema

Now that we know extreme values exist, it's nice to know where to find them. For this, we have the following important theorem.

Theorem (The First Derivative Theorem for Local Extreme Values)

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Finding Extrema

Proof.

Suppose f is a function on $[a, b]$ has a maximum at $x = c$ with $a < c < b$ and $f'(c)$ exists.

For $a < x < c$, we have $f(x) \leq f(c)$ since f has a maximum value at $x = c$. So,

$$f(x) - f(c) \leq 0.$$

We also have $x - c < 0$. So,

$$\frac{f(x) - f(c)}{x - c} \geq 0,$$

so,

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'_-(c) \geq 0.$$

Finding Extrema

Proof (cont.)

For $c < x < b$, we have $f(x) \leq f(c)$ since f has a maximum value at $x = c$. So,

$$f(x) - f(c) \leq 0.$$

We also have $x - c > 0$. So,

$$\frac{f(x) - f(c)}{x - c} \leq 0,$$

so,

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = f'_+(c) \leq 0.$$

Since $f'(c)$ exists, these two one-sided limits must be equal, which forces $f'(c) = 0$.

A similar proof works if f has a minimum value at $x = c$.

Finding Extrema

What this says is that relative extrema can only occur at one of these places:

1. Where the derivative is zero.
2. Where the derivative is undefined.
3. At endpoints of an interval.

Critical Points

Definition

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

With this definition, relative extrema can occur at one of these places:

1. At the critical points.
2. At endpoints of an interval.

Steps to Finding Absolute Extrema

Finding the Absolute Extreme of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f inside the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

Example 1

Example

Find the absolute maximum and the absolute minimum for the function $f(x) = 6x^2 - x^3$ on the interval $[-1, 5]$

Solution

We first compute the derivative.

$$f'(x) = 12x - 3x^2 = 3x(4 - x)$$

This derivative is defined everywhere and is zero at $x = 0$ *and* $x = 4$. *Both of these are inside the given interval.*

Example 1

Solution

We form a T-chart:

x	$f(x)$
-1	7
0	0
4	32
5	25

From this, we see the maximum value is 32 and occurs at $x = 4$ *and the minimum value is 0 and occurs at* $x = 0$.

Example 2

Example

Find the absolute maximum and minimum values of the function $g(x) = \sqrt{4 - x^2}$ on the interval $-2 \leq x \leq 1$.

Solution

First, we take the derivative:

$$g'(x) = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{4 - x^2}}.$$

The derivative is zero at $x = 0$ and the derivative is undefined at $x = \pm 2$, but only $x = -2$ is inside the domain.

Example 2

Solution

We now throw in the endpoints and create a table:

x	$f(x)$
-2	0
0	2
1	$\sqrt{3}$

From this we see that the absolute minimum is 0 and occurs at $x = -2$ *and the absolute maximum is 2 and occurs at* $x = 0$.

Example 3

Example

Find the critical points and domain endpoints for the function $y = x\sqrt{4 - x^2}$. Then find the value of the function at each of these points and identify extreme values (absolute and local).

Solution

Since the argument of the square root function must be nonnegative, domain of this function is the set of all x so that $4 - x^2 \geq 0$. We compute

$$
\begin{aligned}
x^2 &\leq 4 \\
|x| &\leq 2 \\
-2 \leq x &\leq 2.
\end{aligned}
$$

So, the domain of the function is the interval $[-2, 2]$.

Example 3

Solution

Next, we take the derivative:

$$\begin{aligned}\frac{dy}{dx} &= (1)(4-x^2)^{1/2} + x \cdot \frac{1}{2}(4-x^2)^{-1/2} \cdot (-2x) \\ &= (4-x^2)^{1/2} - x^2(4-x^2)^{-1/2} \\ &= (4-x^2)^{-1/2}\left[(4-x^2) - x^2\right] = (4-x^2)^{-1/2}(4-2x^2) \\ &= \frac{2(2-x^2)}{\sqrt{4-x^2}}.\end{aligned}$$

The derivative is zero at $x = \pm\sqrt{2}$ and the derivative is undefined at $x = \pm 2$.

Example 3

Solution

We now throw in the endpoints and create a table:

x	$f(x)$
-2	0
$-\sqrt{2}$	-2
$\sqrt{2}$	2
2	0

From this we see that the absolute minimum is -2 *and occurs at* $x = -\sqrt{2}$ *and the absolute maximum is 2 and occurs at* $x = \sqrt{2}$*. The function has a local minimum of 0 at* $x = 2$ *and a local maximum of 0 at* $x = -2$*.*

(These last two are due to the fact these points are at the endpoints of the domain, so they must be local maxima or local minima.)

Example 4

Example

Find the absolute maximum and minimum values of the function

$$f(x) = \frac{1}{x} + \ln x, \quad \frac{1}{2} \leq x \leq 4,$$

and say where they occur.

Solution

FIrst, we take the derivative:

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x}$$
$$= \frac{x-1}{x^2}.$$

The derivative is zero at $x = 1$. Notice the function itself is undefined at $x = 0$ so this is not a critical point.

Example 4

Solution

We now throw in the endpoints and create a table:

x	$f(x)$
$\frac{1}{2}$	$2 - \ln 2 \approx 1.307$
1	1
4	$\frac{1}{4} + \ln 4 \approx 1.636$

From this we see that the absolute minimum is 1 and occurs at $x = 1$ *and the absolute maximum is* $\frac{1}{4} + \ln 4$ *and occurs at* $x = 4$.