Inverse Trigonometric Functions

William M. Faucette

University of West Georgia

Outline

Inverse Trigonometric Functions

- As usual, you should read section 3.9 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Recall the inverse sine function and the inverse cosine function.

Inverse Sine and Inverse Cosine

$$y = \sin^{-1} x \quad \text{if } x = \sin y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$
$$y = \cos^{-1} x \quad \text{if } x = \cos y \text{ and } 0 \leq y \leq \pi.$$

Now we introduce the other four inverse trigonometric functions.

The Other Inverse Trigonometric Functions

$$y = \tan^{-1} x \text{ if } x = \tan y \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = \cot^{-1} x \text{ if } x = \cot y \text{ and } 0 < y < \pi$$

$$y = \sec^{-1} x \text{ if } x = \sec y \text{ and } 0 \leq y \leq \pi$$

$$y = \csc^{-1} x \text{ if } x = \csc y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

A word of caution here: Unlike $\sin^{-1} x$ and $\cos^{-1} x$, there is no general agreement on how these functions are defined. These are the definitions we will use.

The Derivative of $\sin^{-1} x$

Let $y = \sin^{-1} x$. Then $\sin y = x$. Taking the derivative implicitly with respect to x, we get

$$\begin{aligned} \frac{d}{dx} \sin y &= \frac{d}{dx} x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

Since $\sin^2 y + \cos^2 y = 1$ and $\sin y = x$, we have $\cos^2 y = 1 - x^2$. Since the angle y is in the first or fourth quadrant, the cosine is positive, so $\cos y = \sqrt{1 - x^2}$. This gives us

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

The Derivative of $\tan^{-1} x$

Let $y = \tan^{-1} x$. Then $\tan y = x$. Taking the derivative implicitly with respect to x, we get

$$\begin{aligned} \frac{d}{dx} \tan y &= \frac{d}{dx} x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y}. \end{aligned}$$

Since $1 + \tan^2 y = \sec^2 y$ and $\tan y = x$, we have $\sec^2 y = 1 + x^2$. This gives us

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

The Derivative of $\sec^{-1} x$

Let $y = \sec^{-1} x$. Then $\sec y = x$. Taking the derivative implicitly with respect to x, we get

$$\begin{aligned} \frac{d}{dx} \sec y &= \frac{d}{dx} x \\ \sec y \tan y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. \end{aligned}$$

The Derivative of $\sec^{-1} x$

If $y = \sec^{-1} x$, then $\sec y = x$ and, using the identity $1 + \tan^2 y = \sec^2 y$, we get $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$.

Since y always lies in either the first or second quadrants, the product $\sec y \tan y$ is always positive. So,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \\ &= \frac{1}{|x|\sqrt{x^2 - 1}}. \end{aligned}$$

Derivatives of the Other Three Inverse Trigonometric Functions

To find the derivatives of the remaining three inverse trigonometric functions, we use these trigonometric identities:

Inverse Trigonometric Identities

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$
$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$
$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x.$$

Derivatives of the Other Three Inverse Trigonometric Functions

Using these identities, we get the derivatives of the cofunctions:

Derivatives of Inverse Cofunctions

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}.$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}.$$

Examples

- $$\frac{d}{dx}\cos^{-1}(x^2) = -\frac{1}{\sqrt{1-(x^2)^2}}\cdot 2x = \frac{-2x}{\sqrt{1-x^4}}$$
- $$\begin{aligned}\frac{d}{dx}\sec^{-1}(2x+1) &= \frac{1}{|2x+1|\sqrt{(2x+1)^2-1}}\cdot 2\\ &= \frac{2}{|2x+1|\sqrt{(2x+1)^2-1}}\end{aligned}$$
- $$\frac{d}{dx}\ln(\tan^{-1}(x)) = \frac{1}{\tan^{-1}(x)}\cdot\frac{1}{1+x^2} = \frac{1}{\tan^{-1}(x)(1+x^2)}$$
- $$\frac{d}{dx}\sin^{-1}(x\sqrt{2}) = \frac{1}{\sqrt{1-\left(x\sqrt{2}\right)^2}}\cdot\sqrt{2} = \frac{\sqrt{2}}{\sqrt{1-2x^2}}$$