Derivatives of Inverse Functions and Logarithms

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Outline

Derivatives of Inverse Functions and Logarithms

- As usual, you should read section 3.8 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Derivatives of Inverses of Differentiable Functions

Recall that a function f has an inverse function g if $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f.

For example, $f(x) = 2x + 4$ and $g(x) = \frac{1}{2}x - 2$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}x - 2\right) = \left(2\left(\frac{1}{2}x - 2\right) + 4\right) = x.$$

$$(g \circ f)(x) = g(f(x)) = g\left(2x + 4\right) = \frac{1}{2}\left(2x + 4\right) - 2 = x.$$

So, f and g are inverse functions.

Notice that $\frac{df}{dx} = 2$ and $\frac{dg}{dx} = \frac{1}{2}$.

Derivatives of Inverses of Differentiable Functions

For another example, $f(x) = x^3 - 2$ and $g(x) = \sqrt[3]{x+2}$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x+2}\right) = \left(\sqrt[3]{x+2}\right)^3 - 2 = x.$$

$$(g \circ f)(x) = g(f(x)) = g\left(x^3 - 2\right) = \sqrt[3]{(x^3-2)+2} = x.$$

So, f and g are inverse functions.

Notice that $\frac{df}{dx} = 3x^2$ and $\frac{dg}{dx} = \frac{1}{3}(x+2)^{-2/3} = \frac{1}{3(x+2)^{2/3}}$, which is $\frac{1}{3y^2}$ if $y = \sqrt[3]{x+2}$.

Derivatives of Inverses of Differentiable Functions

Let's examine this more closely. Suppose f and g are inverse functions. Then

$$(f \circ g)(x) = f(g(x)). \tag{1}$$

Let $y = g(x)$. Then $x = f(y)$. We take the derivative of Equation 1 and get

$$\begin{aligned} f'(g(x)) \cdot g'(x) &= 1 \\ f'(y) \cdot g'(x) &= 1 \\ g'(x) &= \frac{1}{f'(y)} \\ &= \frac{1}{f'(g(x))}. \end{aligned}$$

This is the relationship between the derivative of a function f and the derivative of its inverse function g.

Derivatives of Inverses of Differentiable Functions

Theorem

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (which is the range of f). The value of $(f^{-1})'$ at a point b, where $b = f(a)$, in the domain of f^{-1} is the reciprocal of the value of $f'(a) = f'(f^{-1}(b))$.

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Example 1

Example

The function $f(x) = x^3$ and its inverse function $f^{-1}(x) = \sqrt[3]{x}$ derivatives $f'(x) = 3x^2$ and $(f^{-1})'(x) = \left(\frac{1}{3}x^{-2/3}\right)$. Let's verify the preceding theorem for these functions.

Solution

By the theorem,

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{f'(\sqrt[3]{x})}\\ &= \frac{1}{3(\sqrt[3]{x})^2}\\ &= \frac{1}{3x^{2/3}}.\end{aligned}$$

Example 2

Example

The function $f(x) = x^2 - 1$, $x \geq 0$, and its inverse function $f^{-1}(x) = \sqrt{x+1}$ derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = \frac{1}{2}(x+1)^{-1/2}$. Let's verify the preceding theorem for these functions.

Solution

By the theorem,

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{f'(\sqrt{x+1})} \\ &= \frac{1}{2\sqrt{x+1}}.\end{aligned}$$

Derivative of the Natural Logarithm Function

Suppose $y = \ln(x)$. Since the natural logarithm and the exponential functions are inverse functions, we have

$$x = e^y.$$

We take the derivative of this implicitly, treating y as a function of x. This gives us

$$\begin{aligned}\frac{d}{dx}x &= \frac{d}{dx}e^y \\ 1 &= e^y\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{e^y} \\ &= \frac{1}{x}.\end{aligned}$$

Derivative of the Natural Logarithm Function

This gives us the following important result:

Derivative of the $\ln(x)$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0.$$

We can extend this result using the Chain Rule. If u is a differentiable function of x, then

Derivative of the $\ln(u)$

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}, \quad u > 0.$$

Example 3

Example

Find $\frac{dy}{dx}$.

1. $y = \dfrac{1}{\ln 3x}$
2. $y = \ln(\sin x)$
3. $y = x \ln \sqrt{x}$

Solution

1. $$\frac{dy}{dx} = \frac{0 \cdot \ln 3x - 1 \cdot \frac{1}{3x} \cdot 3}{(\ln 3x)^2} = \frac{-1}{x(\ln 3x)^2}.$$
2. $$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x.$$
3. $$\frac{dy}{dx} = (1) \ln \sqrt{x} + x \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2} = \ln(\sqrt{x}) + \frac{1}{2}.$$

Example 4

Example

Find the derivative of $y = (x^2 \ln x)^4$.

Solution

We take the derivative using the Power Rule, the Chain Rule, and the Product Rule.

$$\begin{aligned}
\frac{dy}{dx} &= 4(x^2 \ln x)^3 \cdot \left[2x \ln x + x^2 \cdot \frac{1}{x} \right] \\
&= 4(x^2 \ln x)^3 \left(2x \ln x + x \right) \\
&= 4x(x^2 \ln x)^3 \left(2 \ln x + 1 \right).
\end{aligned}$$

Derivative of a^u and $\log_a u$

Suppose $y = a^x$. Then $\ln(y) = \ln(a^x) = x \ln a$, by the Power Rule for logarithms. Raising e to this power gives

$$y = e^{\ln y} = e^{x \ln a}.$$

Taking the derivative using the Chain Rule, we get

$$\begin{aligned} \frac{dy}{dx} &= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \\ &= e^{x \ln a} \cdot \ln a \\ &= a^x \ln a. \end{aligned}$$

Derivative of a^u and $\log_a(u)$

This gives us the following important result:

Derivative of a^x

$$\frac{d}{dx}a^x = a^x \ln a, \quad x > 0.$$

We can extend this result using the Chain Rule:

Derivative of a^u

If u is a differentiable function of x, then a^u is a differentiable function of x wherever $u > 0$, and

$$\frac{d}{dx}a^u = a^u \ln a \cdot \frac{du}{dx}.$$

Derivative of a^u and $\log_a u$

Suppose $y = \log_a x$. Then $a^y = x$ since $\log_a x$ and a^x are inverse functions. Taking the derivative of $a^y = x$, treating y as a differentiable function of x, we get

$$\begin{aligned}
\frac{d}{dx} a^y &= \frac{d}{dx} x \\
a^y \ln a \cdot \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{a^y \ln a} \\
&= \frac{1}{x \ln a}.
\end{aligned}$$

Derivative of a^u and $\log_a(u)$

This gives us the following important result:

Derivative of $\log_a x$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}, \quad x > 0.$$

We can extend this result using the Chain Rule:

Derivative of $\log_a u$

If u is a differentiable function of x, then $\log_a u$ is a differentiable function of x wherever $u > 0$, and

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a} \cdot \frac{du}{dx}$$

Example 5

Example

Find $\frac{dy}{dx}$.

1. $y = 2^x$
2. $y = \log_4 x + \log_4 x^2$
3. $y = 3^{\log_2 x}$

Solution

1. $\dfrac{dy}{dx} = 2^x \ln 2$
2. $\dfrac{dy}{dx} = \dfrac{1}{x \ln 4} + \dfrac{1}{x^2 \ln 4} \cdot 2x = \dfrac{3}{x \ln 4}$
3. $\dfrac{dy}{dx} = 3^{\log_2 x} \ln 3 \cdot \dfrac{1}{x \ln 2} = \dfrac{3^{\log_2 x} \ln 3}{x \ln 2}$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**.

Example 6

Example

Use logarithmic differentiation to find the derivative of $\frac{dy}{dx}$ if $y = \sqrt{(x^2+1)(x-1)^2}$.

Solution

We start by taking the natural logarithm of both sides of the equation and using the rules of logarithms:

$$\begin{aligned}\ln y &= \ln\left(\sqrt{(x^2+1)(x-1)^2}\right) = \frac{1}{2}\ln\left[(x^2+1)(x-1)^2\right]\\ &= \frac{1}{2}\left[\ln(x^2+1)+\ln\left((x-1)^2\right)\right]\\ &= \frac{1}{2}\left[\ln(x^2+1)+2\ln(x-1)\right]\\ &= \frac{1}{2}\ln(x^2+1)+\ln(x-1).\end{aligned}$$

Example 6

Solution

Now, take the derivative implicitly with respect to x:

$$\ln y = \frac{1}{2}\ln(x^2+1) + \ln(x-1)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot(2x) + \frac{1}{x-1}$$
$$= \frac{x}{x^2+1} + \frac{1}{x-1}.$$

Example 6

Solution

Finally, multiply both sides by y to solve for $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \frac{x}{x^2+1} + \frac{1}{x-1} \\
\frac{dy}{dx} &= y\left[\frac{x}{x^2+1} + \frac{1}{x-1}\right] \\
&= \sqrt{(x^2+1)(x-1)^2}\left[\frac{x}{x^2+1} + \frac{1}{x-1}\right] \\
&= \frac{x(x-1)}{\sqrt{x^2+1}} + \sqrt{x^2+1}.
\end{aligned}$$

Irrational Exponents and the Power Rule (General Version)

If we want to define what it means to take a real number $x > 0$ to any real power n, we certainly want $\ln(x^n) = n \ln x$. This motivates the following definition.

Definition

For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

General Power Rule for Derivatives

General Power Rule for Derivatives

For any $x > 0$ and for any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} exist.

General Power Rule for Derivatives

Proof.

If n is any real number, then $x^n = e^{n \ln x}$, by definition. Taking the derivative with respect to x, we get

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} \\ &= e^{n \ln x} \cdot \frac{n}{x} \\ &= x^n \cdot \frac{n}{x} \\ &= n x^{n-1}. \end{aligned}$$

□

Example 7

Example

Use logarithmic differentiation to find the derivative of $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$.

Solution

We start by taking the natural logarithm of both sides of the equation and using the rules fo logarithms:

$$\ln y = \ln\left(\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\right)$$
$$= \ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1).$$

Example 7

Solution

Now, take the derivative implicitly with respect to x:

$$
\begin{aligned}
\ln y &= \ln(x) + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{2}{3}\cdot\frac{1}{x+1} \\
&= \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}
\end{aligned}
$$

Example 7

Solution

Finally, multiply both sides by y to solve for $\frac{dy}{dx}$.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$$

$$= \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$$

$$= \frac{\sqrt{x^2+1}}{(x+1)^{2/3}} + \frac{x^2}{(x+1)^{2/3}\sqrt{x^2+1}} - \frac{2x\sqrt{x^2+1}}{3(x+1)^{5/3}}.$$

The Number e Expressed as a Limit

Earlier in the course, we defined e to be the number so that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

We also noted then that e is between 2 and 3.

The Number e Expressed as a Limit

Theorem

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof.

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$, so $f'(1) = 1$. By the definition of the derivative

$$\begin{aligned} f'(1) &= \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} \\ &= \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \to 0} \ln[(1+x)^{1/x}]. \end{aligned}$$

The Number e Expressed as a Limit

Proof.

But we know that $f'(1) = 1$, so

$$\lim_{x \to 0} \ln[(1+x)^{1/x}] = 1$$

$$\lim_{x \to 0} e^{\ln[(1+x)^{1/x}]} = e^1$$

$$\lim_{x \to 0} (1+x)^{1/x} = e.$$

□