# Implicit Differentiation

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# Outline

# Derivative of a Composite Function

- As usual, you should read section 3.7 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

# Implicitly Defined Functions

Generally our functions are given as $y = f(x)$. The dependent variable $y$ is given explicitly as a function of the independent variable $x$. However, it is also possible to define functions implicitly. As an example, take the equation

$$x^2 + y^2 = 25.$$

This equation can define $y$ as a function of $x$ (actually two functions of $x$) :

$$y = \pm\sqrt{25 - x^2},$$

We would like to be able to compute $\frac{dy}{dx}$ directly without solving the equation for $y$.

# Example 1

## Example

Suppose the equation $x^2y + xy^2 = 6$ defines $y$ as a differentiable function of $x$. Compute $\frac{dy}{dx}$.

## Solution

*The crucial thing to remember here is that there is only one variable. The remaining letters are either constants or mysterious functions of the variable. In this case, $x$ is the variable and $y$ is some function of $x$. Remembering that, we take the derivative of the entire equation.*

$$\left[\frac{d}{dx}(x^2)\cdot y + x^2\cdot\frac{d}{dx}(y)\right] + \left[\frac{d}{dx}(x)\cdot y^2 + x\cdot\frac{d}{dx}(y^2)\right] = \frac{d}{dx}(6)$$

$$\left[2x\cdot y + x^2\cdot\frac{dy}{dx}\right] + \left[(1)\cdot y^2 + x\cdot 2y\cdot\frac{d}{dx}(y)\right] = 0$$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0.$$

# Example 1

## Solution

*Next, you solve this equation for* $\frac{dy}{dx}$.

$$\begin{aligned} 2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(x^2 + 2xy\right) &= -2xy - y^2 \\ \frac{dy}{dx} &= \frac{-2xy - y^2}{x^2 + 2xy} \\ &= -\frac{y(2x + y)}{x(x + 2y)}. \end{aligned}$$

# Example 2

## Example

Find the slope of the tangent line to the curve $x^2 + xy - y^2 = 1$ at the point $(2, 3)$.

## Solution

*First, we compute the derivative implicitly:*

$$2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0.$$

# Example 2

## Solution

*Now we substitute the values we're given and solve for* $\frac{dy}{dx}$.

$$\begin{aligned} 2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} &= 0 \\ 2(2) + (3) + 2\frac{dy}{dx} - 2(3)\frac{dy}{dx} &= 0 \\ 7 - 4\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{7}{4} \end{aligned}$$

*at the point* $(2, 3)$. *This is the slope of the tangent line.*

# Implicit Differentiation

## Implicit Differentiation

1. Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

# Example 3

## Example

Compute $\frac{dy}{dx}$ if $e^{2x} = \sin(x + 3y)$.

## Solution

*First we take the derivative treating y as a function of the variable x:*

$$2e^{2x} = \cos(x + 3y)\left(1 + 3\frac{dy}{dx}\right)$$

$$2e^{2x} = \cos(x + 3y) + 3\cos(x + 3y)\frac{dy}{dx}.$$

# Example 3

## Solution

*Now solve for* $\frac{dy}{dx}$*:*

$$\begin{aligned} 2e^{2x} &= \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx} \\ 2e^{2x} - \cos(x+3y) &= 3\cos(x+3y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}. \end{aligned}$$

# Derivatives of Higher Order

To find the second derivative implicitly, follow the following procedure:

1. Compute $\frac{dy}{dx}$. This gives you the first derivative in terms of $x$ and $y$.
2. Take the next derivative. This gives $\frac{d^2y}{dx^2}$
3. If necessary, substitute the expression for $\frac{dy}{dx}$ obtained from step 1 into the expression for $\frac{d^2y}{dx^2}$ obtained in the last step.
4. Simplify the expression for $\frac{d^2y}{dx^2}$ algebraically.

# Example 4

## Example

Compute $\frac{d^2y}{dx^2}$ if $xy + y^2 = 1$.

## Solution

*We compute the first derivative implicitly:*

$$\begin{aligned} y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-y}{x+2y}. \end{aligned}$$

# Example 4

## Solution

*Now, take the next derivative using the Quotient Rule:*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-y}{x+2y} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}(x+2y) - (-y)(1+2\frac{dy}{dx})}{(x+2y)^2} \\
&= \frac{-\frac{dy}{dx}(x+2y) + y(1+2\frac{dy}{dx})}{(x+2y)^2} \\
&= \frac{-x\frac{dy}{dx} - 2y\frac{dy}{dx} + y + 2y\frac{dy}{dx}}{(x+2y)^2} \\
&= \frac{-x\frac{dy}{dx} + y}{(x+2y)^2}.
\end{aligned}$$

# Example 4

## Solution

*Finally, we substitute* $\frac{dy}{dx}$ *into the expression for* $\frac{d^2y}{dx^2}$ *and simplify.*

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{-x\frac{dy}{dx} + y}{(x+2y)^2} \\
&= \frac{-x\left(\frac{-y}{x+2y}\right) + y}{(x+2y)^2} = \frac{\left(\frac{xy}{x+2y}\right) + y}{(x+2y)^2} \\
&= \frac{xy + y(x+2y)}{(x+2y)^3} \\
&= \frac{2y(x+y)}{(x+2y)^3}.
\end{aligned}$$

# Example 5

## Example

Find the lines that are tangent and normal to the curve

$$x^2 + y^2 = 25$$

at the point $(3, -4)$.

## Solution

*First, we compute* $\frac{dy}{dx}$ *implicitly.*

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}. \end{aligned}$$

*At the point* $(3, -4)$, $\frac{dy}{dx} = \frac{3}{4}$. *This is the slope of the tangent line. The slope of the normal line is the negative reciprocal of this:* $-\frac{4}{3}$.

# Example 5

## Solution

*So, the tangent line to the curve* $x^2 + y^2 = 25$ *at the point* $(3, -4)$ *is*

$$\begin{aligned} y - (-4) &= \frac{3}{4}(x - 3) \\ y &= \frac{3}{4}x - \frac{25}{4} \end{aligned}$$

*and the normal line to the curve* $x^2 + y^2 = 25$ *at the point* $(3, -4)$ *is*

$$\begin{aligned} y - (-4) &= -\frac{4}{3}(x - 3) \\ y &= -\frac{4}{3}x. \end{aligned}$$