Derivatives of Trigonometric Functions

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Outline

Derivatives of Trigonometric Functions

- As usual, you should read section 3.5 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Derivative of the Sine Function

We recall the formulas from precalculus for sine and cosine of a sum:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

We will use these to find the derivative of the sine function and the cosine function.

Derivative of the Sine Function

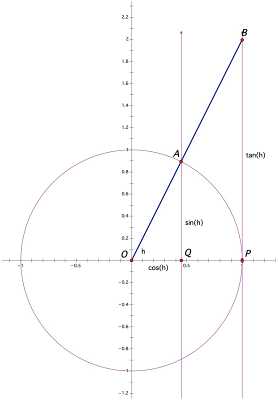
Let $f(x) = \sin x$. To compute the derivative of the sine function, we apply the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{[\sin(x)\cos(h)+\cos(x)\sin(h)]-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right]+\cos(x)\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right].
\end{aligned}$$

We need to compute the two limits that appear here.

Derivative of the Sine Function

Consider the sketch in the figure below. We note that the area of ΔAOQ is less than the area of the sector AOP which in turn is less than the area of ΔBOP. This gives us the inequality on the right.



$$\text{area}\Delta AOQ \leq \text{area}AOP \leq \text{area}\Delta BOP$$

$$\frac{1}{2}\sin(h)\cos(h) \leq \frac{1}{2}h \leq \frac{1}{2}\tan(h)$$

$$\sin(h)\cos(h) \leq h \leq \frac{\sin(h)}{\cos(h)}$$

Figure: Sketch for Limits

Derivative of the Sine Function

For $0 < h < \frac{\pi}{2}$, we have

$$\begin{aligned}
\sin(h)\cos(h) &\leq h \leq \frac{\sin(h)}{\cos(h)} \\
\cos(h) &\leq \frac{h}{\sin(h)} \leq \frac{1}{\cos(h)} \\
\frac{1}{\cos(h)} &\geq \frac{\sin(h)}{h} \geq \cos(h).
\end{aligned}$$

For $-\frac{\pi}{2} < h < 0$, we have

$$\begin{aligned}
\sin(h)\cos(h) &\leq h \leq \frac{\sin(h)}{\cos(h)} \\
\cos(h) &\geq \frac{h}{\sin(h)} \geq \frac{1}{\cos(h)} \\
\frac{1}{\cos(h)} &\leq \frac{\sin(h)}{h} \leq \cos(h).
\end{aligned}$$

Derivative of the Sine Function

So, in either case, $\dfrac{\sin(h)}{h}$ is between $\cos(h)$ and $\dfrac{1}{\cos(h)}$.

Since $\lim\limits_{h\to 0}\cos(h) = \cos(0) = 1$ and $\lim\limits_{h\to 0}\dfrac{1}{\cos(h)} = \dfrac{1}{\cos(0)} = 1$, by the Sandwich Theorem, we have

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1.$$

Derivative of the Sine Function

We use this to compute the other limit:

$$\begin{aligned}
\frac{cos(h)-1}{h} &= \frac{cos(h)-1}{h}\cdot\frac{cos(h)+1}{cos(h)+1}\\
&= \frac{cos^2(h)-1}{h(cos(h)+1)}\\
&= -\frac{1-cos^2(h)}{h(cos(h)+1)}\\
&= -\frac{\sin^2(h)}{h(cos(h)+1)}.
\end{aligned}$$

Derivative of the Sine Function

So,

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos(h)-1}{h} &= \lim_{h\to 0} \left[-\frac{\sin^2(h)}{h(\cos(h)+1)} \right] \\
&= \lim_{h\to 0} \left[-\frac{\sin(h)}{h} \cdot \frac{\sin(h)}{\cos(h)+1} \right] \\
&= \left[-\lim_{h\to 0} \frac{\sin(h)}{h} \right] \cdot \lim_{h\to 0} \left[\frac{\sin(h)}{\cos(h)+1} \right] \\
&= (-1) \cdot \left[\frac{\sin(0)}{\cos(0)+1} \right] \\
&= (-1) \cdot 0 = 0.
\end{aligned}$$

Derivative of the Sine Function

Returning to our original computation, we get

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \sin(x)\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] + \cos(x)\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right]\\ &= \sin(x)(0) + \cos(x)(1)\\ &= \cos(x).\end{aligned}$$

This gives us

The Derivative of $\sin(x)$

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Derivative of the Cosine Function

Let $f(x) = \cos x$. To compute the derivative of the sine function, we apply the definition of the derivative:

$$
\begin{aligned}
\frac{d}{dx}\cos(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{[\cos(x)\cos(h)-\sin(x)\sin(h)]-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)-\sin(x)\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h}\\
&= \cos(x)\lim_{h\to 0}\left[\frac{cos(h)-1}{h}\right]-\sin(x)\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right]\\
&= \cos(x)(0)-\sin(x)(1)\\
&= -\sin(x).
\end{aligned}
$$

Derivative of the Cosine Function

This gives us

Derivatives of $\sin(x)$ and $\cos(x)$

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Derivatives of the Other Trigonometric Functions

To compute the derivatives of the remaining trigonometric functions, we use their definitions and the Quotient Rule.

Recall the definitions of the other four trigonometric functions:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

Derivative of $\tan(x)$

We compute the derivative of $\tan(x)$:

$$
\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}\sin(x)\cdot\cos(x) - \sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}
$$

Derivative of $\cot(x)$

We compute the derivative of $\cot(x)$:

$$\begin{aligned}
\frac{d}{dx}\cot(x) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \frac{\frac{d}{dx}\cos(x)\cdot\sin(x) - \cos(x)\cdot\frac{d}{dx}\sin(x)}{\sin^2(x)} \\
&= \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)} \\
&= \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} \\
&= -\csc^2(x).
\end{aligned}$$

Derivative of $\sec(x)$

We compute the derivative of $\sec(x)$:

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}(1)\cdot\cos(x) - (1)\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{(0)\cdot\cos(x) - (1)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} \\
&= \sec(x)\tan(x).
\end{aligned}$$

Derivative of $\csc(x)$

We compute the derivative of $\csc(x)$:

$$
\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{\frac{d}{dx}(1)\cdot\sin(x) - (1)\frac{d}{dx}\sin(x)}{\sin^2(x)} \\
&= \frac{(0)\cdot\sin(x) - (1)(\cos(x))}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x).
\end{aligned}
$$

Summary of Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Example 1

Example

Find $\frac{dy}{dx}$ if $y = x^2 \cos x$.

Solution

We use the Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) \cdot \cos x + x^2 \cdot \frac{d}{dx}(\cos x) \\ &= 2x \cos x + x^2(-\sin x) \\ &= 2x \cos x - x^2 \sin x.\end{aligned}$$

Example 2

Example

Find $\frac{dy}{dx}$ if $y = \dfrac{\cos x}{1 + \sin x}$.

Solution

We use the Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x) \cdot (1 + \sin x) - \cos x \cdot \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} \\
&= \frac{-\sin x \cdot (1 + \sin x) - \cos x \cdot \cos x}{(1 + \sin x)^2} \\
&= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\
&= \frac{-\sin x - 1}{(1 + \sin x)^2} = -\frac{1 + \sin x}{(1 + \sin x)^2} = -\frac{1}{1 + \sin x}.
\end{aligned}$$

Example 3

Example

Find $\frac{dy}{dx}$ if $y = x^2 \cos x - 2x \sin x - 2 \cos x$.

Solution

We compute:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(x^2 \cos x - 2x \sin x - 2 \cos x\right) \\
&= \frac{d}{dx}\left(x^2 \cos x\right) - \frac{d}{dx}(2x \sin x) - \frac{d}{dx}(2 \cos x) \\
&= \left[2x \cos x + x^2\left(-\sin x\right)\right] - [2 \sin x + 2x \cos x] - 2(-\sin x) \\
&= -x^2 \sin x.
\end{aligned}$$

Example 4

Example

Find $\frac{dy}{dx}$ if $y = (\sec x + \tan x)(\sec x - \tan x)$.

Solution

We compute

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\sec x + \tan x) \cdot (\sec x - \tan x) + \\
&\quad + (\sec x + \tan x) \cdot \frac{d}{dx}(\sec x - \tan x) \\
&= (\sec x \tan x + \sec^2 x) \cdot (\sec x - \tan x) + \\
&\quad + (\sec x + \tan x) \cdot (\sec x \tan x - \sec^2 x) \\
&= \sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \sec^2 x \tan x + \\
&\quad + \sec^2 x \tan x - \sec^3 x + \sec x \tan^2 x - \sec^2 x \tan x \\
&= 0.
\end{aligned}$$

Example 4

Example

Find $\frac{dy}{dx}$ if $y = (\sec x + \tan x)(\sec x - \tan x)$.

Solution

We could have saved ourselves a lot of work by remembering that

$$(\sec x + \tan x)(\sec x - \tan x) = \sec^2 x - \tan^2 x = 1.$$

So, the derivative of this is 0.

Example 5

Example

Graph the curve

$$y = 1 + \cos x,$$

over the interval $-\frac{3\pi}{2} \leq x \leq 2\pi$, together with the tangent lines at $x = -\frac{\pi}{3}$ and $x = \frac{3\pi}{2}$. Label the curve and tangent line with its equation.

Solution

We first compute $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\sin x$$

$$\frac{dy}{dx}\left(-\frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{dy}{dx}\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{3\pi}{2}\right) = 1.$$

Example 5

Solution

At $x = -\frac{\pi}{3}$*, the slope is* $\frac{\sqrt{3}}{2}$ *and the point is* $\left(-\frac{\pi}{3}, \frac{3}{2}\right)$*. The equation of that tangent line is*

$$y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right).$$

At $x = \frac{3\pi}{2}$*, the slope is* 1 *and the point is* $\left(\frac{3\pi}{2}, 1\right)$*. The equation of that tangent line is*

$$y - 1 = x - \frac{3\pi}{2}.$$

The sketch of the curve and the two tangent lines are on the next slide.

Example 5

Solution

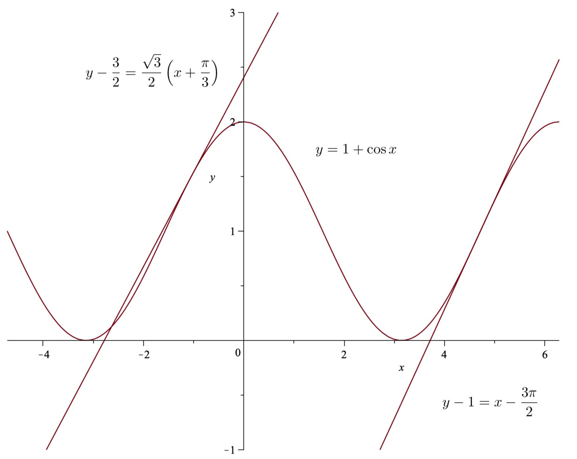


Figure: *Sketch for Example 5*

Example 6

Example

Find all points on the curve $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

Solution

The slope of the line $y = 2x$ is 2, so we have to find all the values of x for $\frac{d}{dx}\tan x = \sec^2 x$ equals 2. We solve

$$\begin{aligned} \sec^2 x &= 2 \\ \sec x &= \sqrt{2} \\ \cos x &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ x &= \pm\frac{\pi}{4}. \end{aligned}$$

Example 6

Solution

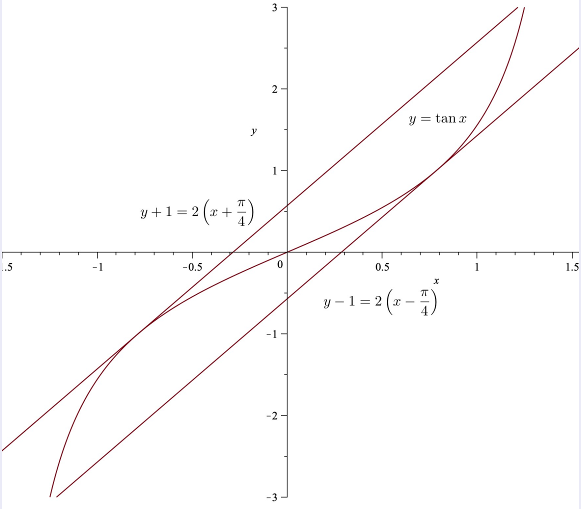


Figure: *Sketch for Example 6*