The Derivative as a Rate of Change

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Outline

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4. Motion Along a Line
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The Derivative as a Rate of Change

- As usual, you should read section 3.4 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Instantaneous Rates of Change

The difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

is the **average rate of change** of f on the interval $[x, x+h]$.

If we then let $h \to 0$, we are taking average rates of change over shorter and shorter intervals. These should be getting closer and closer to the **instantaneous rate of change** of f at x. The instantaneous rate of change of f at x is therefore

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

The interpretation of the derivative as the instantaneous rate of change is important.

Instantaneous Rates of Change

Definition

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

We usually just say "rate of change" rather than "instantaneous rate of change".

Example 1

Example

The area A of a circle is related to its radius by the equation

$$A = \pi r^2.$$

How fast does the area change with respect to the radius when the radius is 5 m?

Solution

The rate of change of the area with respect to the radius is the derivative:

$$\frac{dA}{dr} = 2\pi r.$$

When $r = 5$ m, this gives $10\pi\ m^2/m$.

Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time

$$s = f(t).$$

The **displacement** over a time interval from t to $t + \Delta t$ is the change in position over the time interval, $\Delta s = f(t + \Delta t) - f(t)$

The **average velocity** over the time interval $[t, t + \Delta t]$ is the displacement divided by the time:

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Displacement, Velocity, Speed, Acceleration, and Jerk

If we take the average velocity over shorter and shorter time intervals—that is, if we let Δt go to zero—we get the **velocity (instantaneous velocity)**.

Definition

If a body's position at time t is $s = f(t)$, then the body's **velocity** at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Displacement, Velocity, Speed, Acceleration, and Jerk

Definition

If a body's position at time t is $s = f(t)$, then the body's **speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

Displacement, Velocity, Speed, Acceleration, and Jerk

Definition

If a body's position at time t is $s = f(t)$, then the body's **acceleration** is the rate of change of the velocity with respect to t.

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Displacement, Velocity, Speed, Acceleration, and Jerk

All bodies fall to Earth with constant acceleration due to gravity, g. The displacement of a dropped object is

$$s = \frac{1}{2}gt^2$$

On Earth, $g \approx 32$ ft/s^2 or 9.8 m/s^2.

Example 2

Example

A ball is released from rest at time $t = 0$ s.

a. How many feet does the ball fall in the first 5 s?
b. What is its velocity, speed, and acceleration when $t = 5$ s.

Solution

The distance traveled by the ball under the effects of gravity is $s = \frac{1}{2}gt^2 = \frac{1}{2} \cdot 32t^2 = 16t^2$. *The distance fallen in the first 5 s is then* $s(5) = 16(5)^2 = 400$ *ft.*

Example 2

Solution

We compute the remaining values requested by taking the various derivatives and evaluating at $t = 5$.

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(16t^2) = 32t$$
$$v(5) = 32 \cdot 5 = 160 \ \ ft/s$$
$$|v(5)| = 160 \ \ ft/s$$
$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(32t) = 32$$
$$a(5) = 32 \ \ ft/s^2$$

So, the velocity is 160 ft/s, as is the speed, and the acceleration is 32 ft/s^2.