

Differentiation Rules

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Differentiation Rules

- As usual, you should read section 3.3 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

The Derivative of a Constant

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof.

We compute the derivative from the definition:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} 0 = 0. \end{aligned}$$



The Power Rule

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof.

We compute the derivative from the definition:

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\&= \lim_{h \rightarrow 0} \frac{(x^n + nx^{n-1}h + h^2(\text{stuff})) - x^n}{h} \\&= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + h^2(\text{stuff})}{h} = \lim_{h \rightarrow 0} \frac{h(nx^{n-1} + h(\text{stuff}))}{h} \\&= \lim_{h \rightarrow 0} nx^{n-1} + h(\text{stuff}) = \lim_{h \rightarrow 0} nx^{n-1} + \lim_{h \rightarrow 0} h(\text{stuff}) \\&= nx^{n-1}.\end{aligned}$$

The power rule is actually valid for all real numbers n , not just for positive integers.

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Example 1

$$\textcircled{1} \quad \frac{d}{dx} x^4 = 4x^3.$$

$$\textcircled{2} \quad \frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\textcircled{3} \quad \frac{d}{dx} \frac{1}{x^2} = \frac{d}{dx} x^{-2} = -2x^{-3} = \frac{-2}{x^3}.$$

$$\textcircled{4} \quad \frac{d}{dx} x^{-2/3} = -\frac{2}{3} x^{-5/3} = -\frac{2}{3x^{5/3}}.$$

$$\textcircled{5} \quad \frac{d}{dx} x^{7/5} = \frac{7}{5} x^{2/5}.$$

$$\textcircled{6} \quad \frac{d}{dx} \frac{1}{\sqrt[3]{x}} = \frac{d}{dx} x^{-1/3} = -\frac{1}{3} x^{-4/3} = -\frac{1}{3x^{4/3}}.$$

The Constant Multiple Rule

Derivative Constant Multiple Rule

If f is differentiable function of x and k is a constant, then

$$\frac{d}{dx}(kf) = k \frac{df}{dx}$$

Proof.

We compute the derivative from the definition:

$$\begin{aligned}\frac{d}{dx}(kf) &= \lim_{h \rightarrow 0} \frac{kf(x+h) - kf(x)}{h} = \lim_{h \rightarrow 0} \frac{k[f(x+h) - f(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[k \cdot \frac{f(x+h) - f(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} k \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= k \frac{df}{dx}.\end{aligned}$$

Example 2

- ① If $f(x) = 4x^3$, then

$$\frac{df}{dx} = \frac{d}{dx}4x^3 = 4\frac{d}{dx}x^3 = 4 \cdot 3x^2 = 12x^2.$$

- ② If $f(x) = -g(x)$, with g differentiable, then

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}(-g(x)) = \frac{d}{dx}[(-1) \cdot g(x)] \\ &= (-1)\frac{d}{dx}g(x) = (-1)\frac{dg}{dx} = -\frac{dg}{dx}.\end{aligned}$$

The Sum Rule

The Sum Rule

If f and g are differentiable functions of x , then their sum $f + g$ is a differentiable function and

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}.$$

Proof.

We compute the derivative from the definition:

$$\begin{aligned}\frac{d}{dx}(f + g) &= \lim_{h \rightarrow 0} \frac{(f + g)(x + h) - (f + g)(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - [f(x) + g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x) + g(x + h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} \\ &= \frac{df}{dx} + \frac{dg}{dx}.\end{aligned}$$



The Difference Rule

Combining the Sum Rule and the Constant Multiple Rule, we get the Difference Rule:

The Difference Rule

If f and g are differentiable functions of x , then their sum $f - g$ is a differentiable function and

$$\frac{d}{dx}(f - g) = \frac{df}{dx} - \frac{dg}{dx}.$$

Example 3

Example

Find the derivative of $y = 5x^3 - 3x^5$.

Solution

We compute

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[5x^3 - 3x^5] \\ &= \frac{d}{dx}(5x^3) - \frac{d}{dx}(3x^5) \\ &= 5\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^5) \\ &= 5 \cdot 3x^2 - 3 \cdot 5x^4 \\ &= 15x^2 - 15x^4.\end{aligned}$$

Example 4

Example

Find the derivative of $y = \frac{1}{3x^2} - \frac{5}{2x}$

Solution

We compute

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\frac{1}{3x^2} - \frac{5}{2x} \right] = \frac{d}{dx} \left[\frac{1}{3}x^{-2} - \frac{5}{2}x^{-1} \right] \\ &= \frac{d}{dx} \left(\frac{1}{3}x^{-2} \right) - \frac{d}{dx} \left(\frac{5}{2}x^{-1} \right) \\ &= \frac{1}{3} \frac{d}{dx} (x^{-2}) - \frac{5}{2} \frac{d}{dx} (x^{-1}) \\ &= \frac{1}{3} \cdot (-2x^{-3}) - \frac{5}{2} \cdot (-x^{-2}) \\ &= -\frac{2}{3x^3} + \frac{5}{2x^2}.\end{aligned}$$

The Exponential Rule

When we apply the definition of derivative to the function $f(x) = a^x$ for $a > 0$, $a \neq 1$, we get

$$\begin{aligned}\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \\ &= \lim_{h \rightarrow 0} a^x \cdot \frac{a^h - 1}{h} = a^x \left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right).\end{aligned}$$

So, the derivative of a^x is some constant times a^x .

We note that this constant is $f'(0)$.

Derivatives of Exponential Functions

We examine the limit

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

For $a = 2$, this limit is approximately 0.69.

For $a = 3$, this limit is approximately 1.10.

It makes sense that for some value between 2 and 3, this limit is 1. We define e to be this value. So,

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

Derivatives of Exponential Functions

By our choice of e , we have to following:

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x.$$

The Product Rule

The Product Rule

If f and g are differentiable at x , then so is their product fg , and

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Example 5

Example

Find the derivative of $y = e^x(x^2 + 1)$.

Solution

By the Product Rule, we compute

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^x) \cdot (x^2 + 1) + e^x \cdot \frac{d}{dx}(x^2 + 1) \\ &= e^x(x^2 + 1) + e^x(2x + 0) \\ &= x^2e^x + e^x + 2xe^x \\ &= e^x(x^2 + 2x + 1) \\ &= e^x(x + 1)^2.\end{aligned}$$

The Quotient Rule

The Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then so is their quotient f/g , and

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}.$$

Example 6

Example

Find the derivative of $y = \frac{x^2 - 1}{x^2 + x - 2}$.

Solution

By the Quotient Rule, we compute

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(x^2 - 1) \cdot (x^2 + x - 2) - (x^2 - 1) \cdot \frac{d}{dx}(x^2 + x - 2)}{(x^2 + x - 2)^2} \\ &= \frac{2x(x^2 + x - 2) - (x^2 - 1) \cdot (2x + 1)}{(x^2 + x - 2)^2} \\ &= \frac{2x^3 + 2x^2 - 4x - (2x^3 + x^2 - 2x - 1)}{(x^2 + x - 2)^2} \\ &= \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2} = \frac{(x - 1)^2}{(x + 2)^2(x - 1)^2} = \frac{1}{(x + 2)^2}.\end{aligned}$$

Products and Quotients

Proof of the Quotient Rule.

We compute

$$\begin{aligned}\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \cdot \frac{g(x+h)g(x)}{g(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x)}{hg(x+h)g(x)} + \frac{f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x)}{hg(x+h)g(x)} + \frac{f(x)[g(x) - g(x+h)]}{hg(x+h)g(x)}\end{aligned}$$

Products and Quotients

Proof of the Quotient Rule.

Continuing...

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x)}{hg(x+h)g(x)} + \frac{f(x)[g(x) - g(x+h)]}{hg(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)-f(x)}{h} \cdot g(x)}{g(x+h)g(x)} - \frac{f(x) \cdot \frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \\ &= \frac{\lim_{h \rightarrow 0} \left[\frac{f(x+h)-f(x)}{h} \right] \lim_{h \rightarrow 0} g(x)}{\lim_{h \rightarrow 0} g(x+h) \lim_{h \rightarrow 0} g(x)} \\ & \quad - \frac{\lim_{h \rightarrow 0} f(x) \cdot \lim_{h \rightarrow 0} \left[\frac{g(x)-g(x+h)}{h} \right]}{\lim_{h \rightarrow 0} g(x+h) \lim_{h \rightarrow 0} g(x)} \\ &= \frac{\frac{df}{dx} \cdot g(x)}{g(x) \cdot g(x)} - \frac{f(x) \cdot \frac{dg}{dx}}{g(x) \cdot g(x)} = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{g(x)^2}. \end{aligned}$$



Example 7

Example

Find the derivative of $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$.

Solution

Using the Quotient Rule, we compute

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(1) \cdot (x^2 - 1)(x^2 + x + 1) - 1 \cdot \frac{d}{dx} [(x^2 - 1)(x^2 + x + 1)]}{[(x^2 - 1)(x^2 + x + 1)]^2} \\ &= \frac{(0) \cdot (x^2 - 1)(x^2 + x + 1) - \frac{d}{dx} [(x^2 - 1)(x^2 + x + 1)]}{(x^2 - 1)^2(x^2 + x + 1)^2} \\ &= \frac{-\frac{d}{dx} [(x^2 - 1)(x^2 + x + 1)]}{(x^2 - 1)^2(x^2 + x + 1)^2}\end{aligned}$$

Example 7

Solution

Now using the Product Rule, we continue

$$\begin{aligned} & -\frac{d}{dx} [(x^2 - 1)(x^2 + x + 1)] \\ & \frac{(x^2 - 1)^2(x^2 + x + 1)^2}{(x^2 - 1)^2(x^2 + x + 1)^2} \\ & = \frac{-\left[\frac{d}{dx}(x^2 - 1) \cdot (x^2 + x + 1) + (x^2 - 1) \cdot \frac{d}{dx}(x^2 + x + 1)\right]}{(x^2 - 1)^2(x^2 + x + 1)^2} \\ & = \frac{-\left[2x(x^2 + x + 1) + (x^2 - 1)(2x + 1)\right]}{(x^2 - 1)^2(x^2 + x + 1)^2} \\ & = \frac{-\left[2x^3 + 2x^2 + 2x + (2x^3 + x^2 - 2x - 1)\right]}{(x^2 - 1)^2(x^2 + x + 1)^2} \\ & = \frac{-\left[2x^3 + 2x^2 + 2x + (2x^3 + x^2 - 2x - 1)\right]}{(x^2 - 1)^2(x^2 + x + 1)^2} \\ & = \frac{-4x^3 - 3x^2 + 1}{(x^2 - 1)^2(x^2 + x + 1)^2}. \end{aligned}$$

Example 8

Example

Find the derivative of $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$.

Solution

$$\frac{dy}{dx}$$

$$= \frac{\frac{d}{dx} [(x+1)(x+2)] \cdot (x-1)(x-2) - (x+1)(x+2) \cdot \frac{d}{dx} [(x-1)(x-2)]}{[(x-1)(x-2)]^2}$$

Example 8

Solution

Now, we need to use the Product Rule twice. Here's the first computation:

$$\begin{aligned}\frac{d}{dx} [(x+1)(x+2)] &= \frac{d}{dx}(x+1) \cdot (x+2) + (x+1) \cdot \frac{d}{dx}(x+2) \\ &= (1) \cdot (x+2) + (x+1) \cdot (1) \\ &= 2x + 3.\end{aligned}$$

Example 8

Solution

Here's the second computation:

$$\begin{aligned}\frac{d}{dx} [(x-1)(x-2)] &= \frac{d}{dx}(x-1) \cdot (x-2) + (x-1) \cdot \frac{d}{dx}(x-2) \\ &= (1) \cdot (x-2) + (x-1) \cdot (1) \\ &= 2x - 3.\end{aligned}$$

Example 8

Solution

Putting all this together, we get

$$\begin{aligned} & \frac{dy}{dx} \\ &= \frac{\frac{d}{dx} [(x+1)(x+2)] \cdot (x-1)(x-2) - (x+1)(x+2) \cdot \frac{d}{dx} [(x-1)(x-2)]}{[(x-1)(x-2)]^2} \\ &= \frac{(2x+3) \cdot (x-1)(x-2) - (x+1)(x+2) \cdot (2x-3)}{(x-1)^2(x-2)^2} \\ &= \frac{2x^3 - 3x^2 - 5x + 6 - (2x^3 + 3x^2 - 5x - 6)}{(x-1)^2(x-2)^2} \\ &= \frac{-6x^2 + 12}{(x-1)^2(x-2)^2} = \frac{-6(x^2 - 2)}{(x-1)^2(x-2)^2}. \end{aligned}$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is differentiable, then $f'(x)$ is a function. If it is also differentiable, we can take its derivative to get $f''(x)$ —the **second derivative** of f . It is written several different ways:

$$f''(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

I will mostly be using $f''(x)$ or y'' or $\frac{d^2y}{dx^2}$.

Example 9

Example

If $y = 2x^3 - 3x^2$, then

$$y' = 6x^2 - 6x$$

$$y'' = 12x - 6$$

$$y''' = 12$$

$$y^{(4)} = 0.$$

Notice that once we get past the first few derivatives, the notation for higher derivatives is a superscript consisting of an arabic (or roman) numeral inside parentheses. The **n th derivative of y with respect to x** is denoted

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y.$$