The Derivative as a Function

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Outline

The Derivative as a Function

- As usual, you should read section 3.2 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

The Derivative as a Function

In the last slideshow we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Now, we apply this definition to a general point x. This makes the derivative a function of x.

Definition (Definition of the Derivative)

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The Derivative as a Function

If f' exists at a particular x, we say that f is **differentiable (has a derivative) at** x. If f' exists at every point in the domain of f, we say f is **differentiable**.

In the definition of the derivative, if we let $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x.

Definition (Alternative Definition of the Derivative)

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x},$$

provided the limit exists.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f(x)$.

Here are two more examples in which we allow x to be any point in the domain of f.

Example 1

Example

Find the derivative of the function

$$f(x) = \frac{1-x}{2x}.$$

Solution

We compute

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1-(x+h)}{2(x+h)} - \frac{1-x}{2x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{1-(x+h)}{2(x+h)} - \frac{1-x}{2x}}{h} \cdot \frac{2x(x+h)}{2x(x+h)} \\ &= \lim_{h\to 0} \frac{(1-x-h)x-(1-x)(x+h)}{2xh(x+h)}. \end{aligned}$$

Example 1

Solution

We continue

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(1-x-h)x-(1-x)(x+h)}{2xh(x+h)} \\ &= \lim_{h\to 0} \frac{(x-x^2-xh)-(x+h-x^2-xh)}{2xh(x+h)} \\ &= \lim_{h\to 0} \frac{-h}{2xh(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{2x(x+h)} \\ &= -\frac{1}{2x^2}. \end{aligned}$$

Example 1

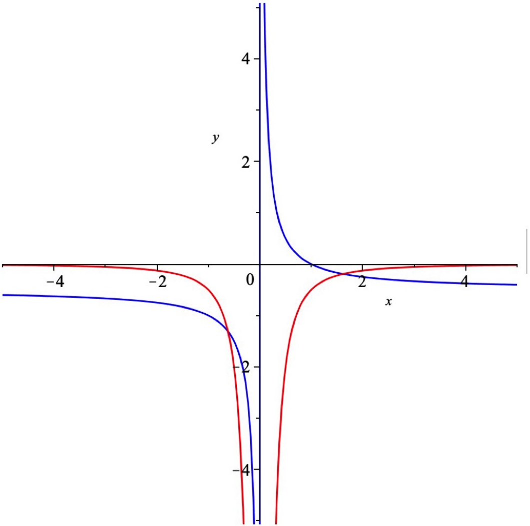

Figure: y=f(x), y=f'(x)

Example 2

Example

Differentiate the function

$$f(x) = x + \frac{9}{x}$$

and find the slope of the tangent line at $x = -3$.

Solution

We compute

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x + h + \frac{9}{x+h} - \left(x + \frac{9}{x}\right)}{h} \\ &= \lim_{h\to 0} \frac{x + h + \frac{9}{x+h} - \left(x + \frac{9}{x}\right)}{h} \cdot \frac{x(x+h)}{x(x+h)} \\ &= \lim_{h\to 0} \frac{x(x+h)^2 + 9x - (x^2+9)(x+h)}{hx(x+h)} \end{aligned}$$

Example 2

Solution

Continuing . . .

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{x(x+h)^2 + 9x - (x^2+9)(x+h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{x(x^2+2xh+h^2) + 9x - (x^3+x^2h+9x+9h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{x^3+2x^2h+xh^2+9x-x^3-x^2h-9x-9h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{x^2h+xh^2-9h}{hx(x+h)} = \lim_{h\to 0} \frac{h(x^2+xh-9)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{x^2+xh-9}{x(x+h)} = \frac{x^2-9}{x^2}.
\end{aligned}
$$

The slope of the tangent line at $x=-3$ *is* $\frac{(-3)^2-9}{(-3)^2}=0$.

Example 2

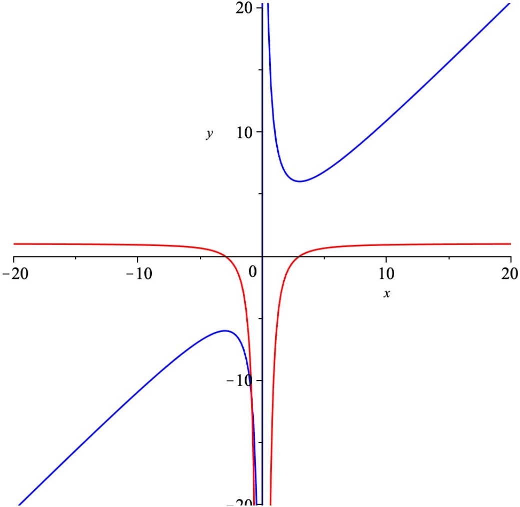

Figure: y=f(x), y=f'(x)

Notation for the Derivative

The derivative of f at x can be denoted by $f'(x)$, f', or $\frac{df}{dx}$. If $y = f(x)$, the derivative of f at x can be denoted $\frac{dy}{dx}$.

As a side note, let's talk about the origin of the last notation. The symbol $\frac{d}{dx}$ is what is called an **operator**. This is the derivative operator and it means "take the derivative of the next expression with respect to x". So,

$$\frac{d}{dx}y$$

means "take the derivative of y with respect to x". It's natural to denote this by $\frac{dy}{dx}$.

The value of the derivative at $x = a$ is denoted

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

Differentiable on an Interval; One-Sided Derivatives

We make the following definitions.

The **right-hand derivative of** f **at** a is

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}.$$

The **left-hand derivative of** f **at** b is

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}.$$

Example 3

Example

Show that $y = |x|$ is differentiable at every point except $x = 0$.

Solution

Suppose $c > 0$. Then we can find a small interval containing c on which x is positive, so $|x| = x$ on this small interval. Then

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{|c+h| - |c|}{h} \\ &= \lim_{h\to 0} \frac{(c+h) - c}{h} \text{ (for } h \text{ small)} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 = 1. \end{aligned}$$

Example 3

Solution

Suppose $c < 0$*. Then we can find a small interval containing* c *on which* x *is negative, so* $|x| = -x$ *on this small interval. Then*

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{|c+h| - |c|}{h} \\ &= \lim_{h\to 0} \frac{-(c+h) - (-c)}{h} \text{ (for } h \text{ small)} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0} -1 = -1. \end{aligned}$$

Example 3

Solution

For $c = 0$ *we compute*

$$
\begin{aligned}
f'(0^+) &= \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} \\
&= \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \\
&= \lim_{h\to 0^+} 1 \\
&= 1.
\end{aligned}
\qquad
\begin{aligned}
f'(0^-) &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\
&= \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \\
&= \lim_{h\to 0^-} -1 \\
&= -1.
\end{aligned}
$$

Since the left-hand derivative and right-hand derivative are not equal at $x = 0$, *the function* $f(x) = |x|$ *is not differentiable at* $x = 0$.

Example 3

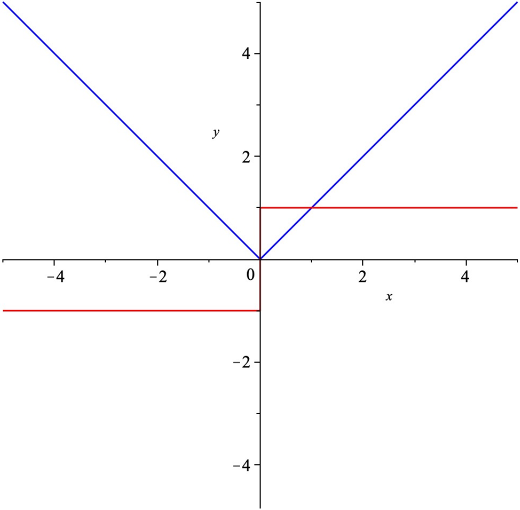

Figure: y=f(x), y=f'(x)

When Does a Function **Not** Have a Derivative at a Point?

There are many reasons why a function may not have a derivative at a given point. The derivative does not exist at the following types of points.

1. A point where the curve comes to a corner.
2. A point where the curve has a cusp. For example $y = x^{2/3}$ has a cusp at $x = 0$.
3. A point where the function has a vertical tangent.
4. A point where the function is discontinuous.
5. A point where the function wildly oscillates.

Differentiable Functions Are Continuous

Theorem (Differentiability Implies Continuity)

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof.

We write

$$f(c+h) = \frac{f(c+h)-f(c)}{h} \cdot h + f(c)$$

Taking the limit as $h \to 0$, we get

$$\begin{aligned}
\lim_{h\to 0} f(c+h) &= \lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h} \cdot h + f(c)\right) \\
&= \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot \lim_{h\to 0} h + \lim_{h\to 0} f(c) \\
&= f'(c)\cdot 0 + f(c) = f(c).
\end{aligned}$$

So, f is continuous at $x = c$. ☐