Linearization and Differentials

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Linearization and Differentials

- As usual, you should read section 3.11 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Linearization

The tangent line to the curve $y = f(x)$ at a point $x = a$ is given by

$$y = f(a) + f'(a)(x - a).$$

This line is the graph of the function

$$L(x) = f(a) + f'(a)(x - a).$$

This is the **linearization** of f at a. As long as x is close to a, $L(x)$ is a good approximation for $f(x)$.

Linearization

Definition

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a).$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Example 1

Example

Write a differential formula that estimates the change in volume os a sphere when the radius changes from r_0 to $r_0 + dr$.

Solution

The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

We compute the derivative $\frac{dV}{dr} = 4\pi r^2$.

So, the differential is $dV = 4\pi r^2\,dr$.

If we substitute the point r_0, we get $dV = 4\pi r_0^2\,dr$.

Example 2

Example

Use the preceding computation to approximate the change in volume of a sphere if the radius of a sphere is change from 10 cm to 10.1 cm.

Solution

Here, the change in r is $dr = 10.1 - 10 = 0.1$.

From the last solution, we know that $dV = 4\pi r_0^2\, dr$.

Substituting $r_0 = 10$ and $dr = 0.1$, we get

$$dV = 4\pi r_0^2 dr = 4\pi(10)^2(0.1) = 40\pi \approx 125.7 \text{ cm}^3.$$

Example 3

Example

The radius of a circle is increased from 2.00 m to 2.02 m. Estimate the resulting change in area.

Solution

The area of a circle is given by $A = \pi r^2$.

Taking the derivative, we get $\frac{dA}{dr} = 2\pi r$.

So, the differential equation is $dA = 2\pi r\, dr$.

We are given that $r = 2$ and $dr = 2.02 - 2 = 0.02$.

Substituting the values we're given, we compute

$$dA = 2\pi r dr = 2\pi(2)(0.02) = 0.08\pi \approx 0.25 \text{ m}^2.$$

Definition

Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx$$

Example 4

Example

If $y = \frac{2x}{1+x^2}$, find dy.

Solution

We take the derivative and abuse the notation:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(2x)'(1+x^2) - 2x(1+x^2)'}{(1+x^2)^2} \\
&= \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} \\
&= \frac{2-2x^2}{(1+x^2)^2} \\
dy &= \frac{2-2x^2}{(1+x^2)^2}\, dx.
\end{aligned}$$

Geometric Meaning of Differential

Suppose $f(x)$ is differentiable, $y = f(x)$, and L is the standard linear approximation with center $x = a$. If we change x by a small amount, $dx = \Delta x$, the corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change on the tangent line is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= L(a + dx) - L(a)\\ &= f(a) + f'(a)((a + dx) - a) - f(a)\\ &= f'(a)dx.\end{aligned}$$

So, the differential dy is just the change in the value on the tangent line.

Differentials

We sometimes write

$$df = f'(x)dx$$

in place of $dy = f'(x)dx$, calling df the **differential of f**.
This notation reminds us of the convenience of the Leibniz notation for a derivative.

$$f'(x) = \frac{df}{dx}$$

so, abusing the notation, we have

$$df = f'(x)dx.$$

Example 5

Example

Find *dy* if

$$y = \sin(5\sqrt{x}).$$

Solution

We compute the derivative and abuse the notation:

$$\begin{aligned} \frac{dy}{dx} &= \cos(5\sqrt{x}) \cdot \frac{d}{dx}(5\sqrt{x}) \\ &= \cos(5\sqrt{x}) \cdot \frac{5}{2}x^{-1/2} \\ &= \frac{5\cos(5\sqrt{x})}{2\sqrt{x}}. \\ dy &= \frac{5\cos(5\sqrt{x})}{2\sqrt{x}}\,dx. \end{aligned}$$

Estimating with Differentials

Suppose we know the value of a differentiable function f at $x = a$ as well as the value of the derivative of f at $x = a$. If we change a to a nearby point $a + dx$, then the actual change in the function

$$f(a + dx) - f(a) = \Delta y$$

is approximated by the differential dy where $y = f(x)$:

$$dy = f'(a)\,dx.$$

So,

$$f(a + dx) \approx f(a) + dy = f(a) + f'(a)\,dx.$$

Hence, we can use the values of $f(a)$, $f'(a)$, and dx to approximate $f(a + dx)$.

Example 6

Example

Write a differential formula that estimates the given change in volume $V = x^3$ of a cube when the edge length change from x_0 to $x_0 + dx$.

Solution

We compute the derivative

$$\begin{aligned} \frac{dV}{dx} &= 3x^2 \\ dV &= 3x^2\,dx \\ dV &= 3x_0^2\,dx. \end{aligned}$$

The change in the volume is approximated by $3x_0^2\,dx$.

Example 7

Example

The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-sectional area?

Solution

Let C be the circumference of the tree, d the diameter of the tree, and A the cross-sectional area of the tree.
The relationship between the circumference and the diameter of the tree is

$$C = \pi\, d$$

so

$$dC = \pi\, dd.$$

If the circumference increases by 2 in, then $dC = 2$ and $dd = dC/\pi = 2/\pi \approx 0.64$ in.

Example 7

Solution

The relationship between the circumference of the tree and the area of the cross-sectional area of the tree is

$$A = \pi \left(\frac{C}{2\pi} \right)^2 = \frac{1}{4\pi} C^2.$$

so

$$dA = \frac{1}{2\pi} C \, dC.$$

When the diameter is 10 in, the circumference of the tree is 10π *in. If the circumference increases by 2 in, then* $dC = 2$ *and* $dA = \frac{1}{2\pi}(10\pi)(2) = 10 \ in^2$.

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x. Then the actual change in the function is

$$\Delta f = f(a + \Delta x) - f(a)$$

and the differential change is

$$df = f'(x)\Delta x$$

What is the error here in the approximation?

Error in Differential Approximation

By definition

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

Let

$$\varepsilon = \frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a).$$

Then $\varepsilon \to 0$ as $\Delta x \to 0$.

Notice that

$$f(a + \Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x.$$

Error in Differential Approximation

Recalling that $\Delta f = f(a + \Delta x) - f(a)$ and $df = f'(a)\Delta x$, we have

$$\begin{aligned} f(a+\Delta x) - f(a) &= f'(a)\Delta x + \varepsilon\Delta x \\ \Delta f &= df + \varepsilon\Delta x \\ \Delta f - df &= \varepsilon\Delta x. \end{aligned}$$

The error between the change in f and the differential equals $\varepsilon\Delta$.

Error in Differential Approximation

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x \tag{1}$$

in which $\varepsilon \to 0$ as $\Delta x \to 0$.

Another Look at the Proof of the Chain Rule

Suppose y is a differentiable function of u at $u = u_0$.

Referring to Equation 1, there exists a function ε_1 so that

$$\Delta y = (f'(u_0) + \varepsilon_1)\Delta u$$

and $\varepsilon_1 \to 0$ as $\Delta u \to 0$.

Another Look at the Proof of the Chain Rule

Suppose u is a differentiable function of x at $x = x_0$.

Referring to Equation 1, there exists a function ε_2 so that

$$\Delta u = (u'(x_0) + \varepsilon_2)\Delta x$$

and $\varepsilon_2 \to 0$ as $\Delta x \to 0$.

Also note that as $\Delta x \to 0$, $\Delta u \to 0$ as well.

Another Look at the Proof of the Chain Rule

Let $u = u(x)$ $u_0 = u(x_0)$.
Then

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\ &= (f'(u_0) + \varepsilon_1)(u'(x_0) + \varepsilon_2). \end{aligned}$$

Taking the limit as $\Delta x \to 0$, we get

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[(f'(u_0) + \varepsilon_1)(u'(x_0) + \varepsilon_2)\right] \\ &= \left[\lim_{\Delta x \to 0}(f'(u_0) + \varepsilon_1)\right]\left[\lim_{\Delta x \to 0}(u'(x_0) + \varepsilon_2)\right] \\ &= \left[\lim_{\Delta u \to 0}(f'(u_0) + \varepsilon_1)\right]\left[\lim_{\Delta x \to 0}(u'(x_0) + \varepsilon_2)\right] \\ &= f'(u_0)u'(x_0) \\ &= f'(u(x_0))u'(x_0). \end{aligned}$$