Related Rates

William M. Faucette

University of West Georgia

Outline

Related Rates

- As usual, you should read section 3.10 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Related Rates Equations

Suppose we are filling a spherical balloon with air. At any particular time, we can compute the volume and radius of the balloon. We also have the relationship

$$V = \frac{4}{3}\pi r^3. \tag{1}$$

As we remarked above, we can treat both V and r as functions of t.

Related Rates Equations

Suppose both V and r are differentiable with respect to t. Then we can take get the derivative of Equation 1 with respect to t to get

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

This gives a relationship between $\frac{dV}{dt}$, the rate at which the volume of the balloon is changing; $\frac{dr}{dt}$, the rate at which the radius is changing; and the radius of the balloon, r.

This is an example of a **related rate equation**, since it relates the rates of change to one another.

Example 1

Example

If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m, at what rate does the cube's

a. surface area change?
b. volume change?

Solution

If x is the edge length of the cube, then the surface area and volume of the cube are given by

a. $S = 6x^2$
b. $V = x^3$,

respectively.

Example 1

Solution

The surface area of the cube is given by the equation $S = 6x^2$. *If we treat both* S *and* x *as differentiable functions of time* t *and take the derivative (with respect to* t*), we get*

$$\frac{dS}{dt} = 12x\frac{dx}{dt}.$$

We are told the edge length of the cube decreases at a rate of 5 m/min, so $\frac{dx}{dt} = -5$ *m/min. So, where* $x = 3$ *m, we compute*

$$\frac{dS}{dt} = 12x\frac{dx}{dt} = 12 \cdot 3 \cdot (-5) = -180.$$

So, the surface area is decreasing at a rate of 180 m^2*/min.*

Example 1

Solution

The volume of the cube is given by the equation $V = x^3$*. If we treat both* V *and* x *as differentiable functions of time* t *and take the derivative (with respect to* t*), we get*

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}.$$

We are told the edge length of the cube decreases at a rate of 5 m/min, so $\frac{dx}{dt} = -5$ *m/min. So, where* $x = 3$ *m, we compute*

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3 \cdot 3^2 \cdot (-5) = -135.$$

So, the volume is decreasing at a rate of 135 m^3*/min.*

Related Rates Problem Strategy

1. *Read the problem.* You should read the problem several times. Try to put it into your own words so you know what it is saying and what it is asking.
2. *Draw a picture and name the variables and constants.* Use t for time. Assume that all variables are differentiable functions of t.
3. *Write down the numerical information* Write down what you know and what you are looking for in terms of the variables or derivatives of the variables.
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

Example 2

Example

The length ℓ of a rectangle is decreasing at the rate of 2 cm/s while the width w is increasing at the rate of 2 cm/s. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of **(a)** the area, **(b)** the perimeter, and **(c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

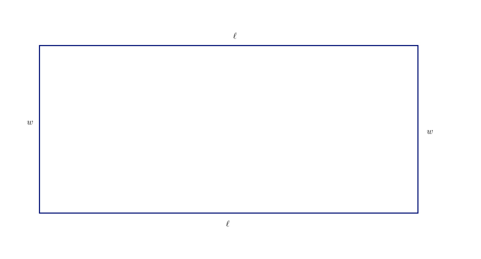

Figure: Sketch for Example 2 with w and ℓ labeled

Example 2

Solution

We are told that the length ℓ of a rectangle is decreasing at the rate of 2 cm/s and the width w of a rectangle is increasing at the rate of 2 cm/s. So,

$$\frac{d\ell}{dt} = -2, \quad \frac{dw}{dt} = 2.$$

We are told that $\ell = 12$ and $w = 5$ at the instant in which we are interested.

Example 2

Solution

The area of the rectangle is given by

$$A = \ell w.$$

Taking the derivative with respect to time t and substituting what we know, we find

$$\begin{aligned} \frac{dA}{dt} &= \frac{d\ell}{dt} w + \ell \frac{dw}{dt} \\ &= (-2)(5) + (12)(2) \\ &= 14. \end{aligned}$$

So, the area is increasing at 14 cm^2/s.

Example 2

Solution

The perimeter of the rectangle is given by

$$P = 2\ell + 2w.$$

Taking the derivative with respect to time t and substituting what we know, we find

$$\begin{aligned} \frac{dP}{dt} &= 2\frac{d\ell}{dt} + 2\frac{dw}{dt} \\ &= 2(-2) + 2(2) \\ &= 0. \end{aligned}$$

So, the perimeter is not changing at this moment.

Example 2

Solution

The length of the diagonal of the rectangle is given by

$$d^2 = \ell^2 + w^2. \tag{2}$$

Using Equation 2 and the values we are given for ℓ and w, we compute

$$\begin{aligned} d^2 &= \ell^2 + w^2 \\ &= (12)^2 + (5)^2 \\ &= 169, \end{aligned}$$

So, $d = 13$ at the moment in which we are interested.

Example 2

Solution

The length of the diagonal of the rectangle is given by

$$d^2 = \ell^2 + w^2.$$

Taking the derivative with respect to time t and substituting what we know, we find

$$\begin{aligned} 2d\frac{dd}{dt} &= 2\ell\frac{d\ell}{dt} + 2w\frac{dw}{dt} \\ 2(13)\frac{dd}{dt} &= 2(12)(-2) + 2(5)(2) \\ 26\frac{dd}{dt} &= -28 \\ \frac{dd}{dt} &= -\frac{28}{26} = -\frac{14}{13}. \end{aligned}$$

So, the length of the diagonal is decreasing at $\frac{14}{13}$ *cm/s.*

Example 3

Example

A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/s.

1. How fast is the top of the ladder sliding down the wall then?
2. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
3. At what rate is the angle θ between the ladder and the ground changing then?

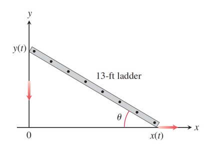

Figure: Sketch for Example 3

Example 3

Solution

In this problem, the picture is already drawn and the variables are already labeled for us. The function $x(t)$ is the distance from the foot of the ladder to the building. The function $y(t)$ is the distance from the top of the ladder to the ground. Since the ladder is 13 feet long, the Pythagorean theorem gives us

$$x^2 + y^2 = 13^2 = 169. \tag{3}$$

We are told that $\frac{dx}{dt} = 5$ when $x = 12$. From Equation 3, we find that when $x = 12$, $y = 5$.

The rate at which the top of the ladder is sliding down the wall is $\frac{dy}{dt}$. This is what we want to find.

Example 3

Solution

Taking the derivative with respect to t, *we get*

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Substituting $\frac{dx}{dt} = 5$, $x = 12$, *and* $y = 5$, *and solving for* $\frac{dy}{dt}$, *we get*

$$\begin{aligned} 2(12)(5) + 2(5)\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -12. \end{aligned}$$

So, the top of the ladder is sliding down the wall at 12 ft/s.

Example 3

Solution

The area of the triangle formed by the ladder, the ground, and the building is

$$A = \frac{1}{2}xy.$$

Taking the derivative with respect to t, *we get*

$$\frac{dA}{dt} = \frac{1}{2}\frac{dx}{dt}y + \frac{1}{2}x\frac{dy}{dt}.$$

We substitute $\frac{dx}{dt} = 5$, $x = 12$, *and* $y = 5$, *from the problem and* $\frac{dy}{dt} = -12$ *from our first computation. Solving for* $\frac{dA}{dt}$, *we get*

$$\frac{dA}{dt} = \frac{1}{2}(5)(5) + \frac{1}{2}(12)(-12) = \frac{25}{2} - \frac{7}{2} = -\frac{119}{2}.$$

So, the area of the triangle is decreasing at a rate of $\frac{119}{2} = 59.5\ ft^2/s$.

Example 3

Solution

If θ *is the angle between the base of the ladder and the ground, then*

$$\tan\theta = \frac{y}{x}.$$

We note that when $x = 12$*, and* $y = 5$, $\tan\theta = \frac{5}{12}$ *and* $\sec\theta = \frac{13}{12}$*. (Draw a triangle!)*

Example 3

Solution

Taking the derivative with respect to t *and substituting* $\frac{dx}{dt} = 5, \frac{dy}{dt} = -12$, $x = 12$, $y = 5$, *and* $\sec\theta = \frac{13}{12}$, *and solving for* $\frac{d\theta}{dt}$, *we get*

$$\begin{aligned}
\sec^2\theta \frac{d\theta}{dt} &= \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2} \\
\left(\frac{13}{12}\right)^2 \frac{d\theta}{dt} &= \frac{(12)(-12) - (5)(5)}{(12)^2} \\
\left(\frac{13}{12}\right)^2 \frac{d\theta}{dt} &= -\frac{169}{144} \\
\frac{d\theta}{dt} &= -1.
\end{aligned}$$

So, at the instant in question, the angle θ *is decreasing at a rate of* 1 *rad/s.*

Example 4

Example

Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.

Example 4

Solution

We have sketched the conical pile of sand in the figure below. We have also labeled as variables the radius r and height h of the pile. Let V be the volume of the cone.

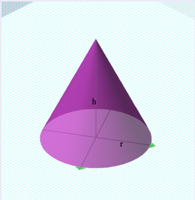

Figure: *Sketch for Example 4*

We are told that the volume is increasing at a rate of 10 m^3/min, so $\frac{dV}{dt} = 10$. We are also told the height of the pile is always three-eighths of the base diameter, so $h = \frac{3}{8} \cdot 2r = \frac{3}{4}r$. When the pile is 4 m high, $4 = \frac{3}{4}r$, so $r = \frac{16}{3}$.

Example 4

Solution

The volume of the cone is $V = \frac{\pi}{3} r^2 h$. *We substitute* $h = \frac{3}{4} r$ *into this equation to get*

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} r^2 \left(\frac{3}{4} r \right) = \frac{\pi}{4} r^3.$$

Taking the derivative of this equation with respect to t *gives us*

$$\frac{dV}{dt} = \frac{\pi}{4} 3r^2 \frac{dr}{dt} = \frac{3\pi}{4} r^2 \frac{dr}{dt}.$$

Example 4

Solution

We have the related rate equation

$$\frac{dV}{dt} = \frac{3\pi}{4} r^2 \frac{dr}{dt}.$$

Substituting $\frac{dV}{dt} = 10$ *and* $r = \frac{16}{3}$ *into this equation and solving, we get*

$$10 = \frac{3\pi}{4}\left(\frac{16}{3}\right)^2 \frac{dr}{dt} = \frac{64\pi}{3}\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{15}{32\pi} \approx 0.1492 \ m/min.$$

Example 4

Solution

Taking the derivative of the equation $h = \frac{3}{4}r$ *with respect to* t, *we get*

$$\frac{dh}{dt} = \frac{3}{4}\frac{dr}{dt}.$$

Substituting $\frac{dr}{dt} = \frac{15}{32\pi}$, *we get*

$$\frac{dh}{dt} = \frac{3}{4} \cdot \frac{15}{32\pi} = \frac{45}{128\pi} \approx 0.1119.$$

So, the radius of the pile is increasing at a rate of approximately 14.92 cm/min. The height of the pile is increasing at a rate of approximately 11.19 cm/min.

Example 5

Example

A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

Solution

We really don't need a picture of a sphere.
Let r be the radius of the balloon, V the volume of the balloon, and S the surface area of the balloon. Then

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2.$$

Example 5

Solution

We are told the volume of the balloon is increasing at a rate of 100π *ft*^{3}*/min, so* $\frac{dV}{dt} = 100\pi$*. We are asked to find* $\frac{dr}{dt}$ *and* $\frac{dS}{dt}$ *when* $r = 5$*.*

Taking the derivative of $V = \frac{4}{3}\pi r^3$ *with respect to* t *gives us*

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Substituting $\frac{dV}{dt} = 100\pi$ *and* $r = 5$ *and solving for* $\frac{dr}{dt}$*, we get*

$$100\pi = 4\pi(5)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = 1.$$

So, the radius is changing at the rate of 1 ft/min.

Example 5

Solution

Taking the derivative of $S = 4\pi r^2$ *with respect to* t *gives us*

$$\frac{dS}{dt} = 4\pi \cdot 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}.$$

Substituting $\frac{dr}{dt} = 1$ *and* $r = 5$ *and solving for* $\frac{dS}{dt}$, *we get*

$$\frac{dS}{dt} = 8\pi(5)(1) = 40\pi.$$

So, the surface area is increasing at the rate of 40π ft^2/min.

Example 6

Example

The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $\frac{dx}{dt} = -1$ m/s and $\frac{dy}{dt} = -5$ m/s. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$.

Solution

We don't really need a picture here either. The distance d from the point (x, y) to the origin is $d = \sqrt{x^2 + y^2}$, but we will square this equation to get

$$d^2 = x^2 + y^2.$$

Taking the derivative with respect to t, we get

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Example 6

Solution

We know that $\frac{dx}{dt} = -1$ *m/s,* $\frac{dy}{dt} = -5$ *m/s, and when the particle passes through the point* $(5, 12)$*, we have* $x = 5$*,* $y = 12$*, so* $d = 13$*. Substituting these values and solving for* $\frac{dd}{dt}$ *we get*

$$\begin{aligned} 2d\frac{dd}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ 2(13)\frac{dd}{dt} &= 2(5)(-1) + 2(12)(-5) \\ 26\frac{dd}{dt} &= -130 \\ \frac{dd}{dt} &= -\frac{130}{26} = -5. \end{aligned}$$

So, the distance from the particle to the origin is decreasing at a rate of 5 m/s.